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The Second Report of The Commission[1] on Post-War Plans

The Improvement of Mathematics in Grades 1 to 14

Introduction

THIS REPORT presents suggestions for improving mathematical instruction from the beginning of the elementary school through the last year of junior college. The program throughout these grades is in need of a thoroughgoing reorganization. The arithmetic of the elementary school can be and must be improved. The high school needs to come to grips with its dual responsibility, (1) to provide sound mathematical training for our future leaders of science, mathematics, and other learned fields, and (2) to insure mathematical competence for the ordinary affairs of life to the extent that this can be done for all citizens as a part of a general education appropriate for the major fraction of the high school population. Then, too, the junior college, which has grown up without a well considered design, should now take stock of its valid functions before it enters its second period of rapid expansion. It is reasonable to believe that the greatest advance can be made if teachers of mathematics in the elementary school, in the secondary school, and in the junior college, attack the problem together. At any rate it is sensible because of the essential continuity of mathematical instruction to plan the improvements in any one grade in terms of the total program.

The report presents a series of constructive theses which in spite of appearances are not offered in dogmatic finality. It is hoped that these tentative guides will be widely discussed in order that desirable modifications may be made in a later report. The Commission is seeking, through the cooperative thinking of teachers of mathematics in these grades, to arrive at a set of principles (a blueprint) for building a stronger program in mathematical education. Although the theses are tentative, it should be noted that they stem from the collective experience and thinking of the whole Commission and from a considerable body of pedagogical literature and investigation. Some have, of course, long been advocated by many leaders.

The mental climate with respect to mathematics is at the moment very favorable. Now is the time to put our house

[1] Created by the Board of Directors of the National Council of Teachers of Mathematics. The first report of the Commission appeared in the May 1944 issue of THE MATHEMATICS TEACHER, pages 226–232. Reprints of the first report may be had for 10¢ each postpaid and the second report for 15¢ each postpaid from THE MATHEMATICS TEACHER, 525 W. 120th St., New York 27, N. Y.

in order by making the improvements that have been postponed for too long. In the postwar period we should be ready to provide the very finest mathematics program for every type of youngster in our schools. It is to that end that the Commission submits the following theses for careful study.[2] The first one is of vital concern to every teacher of mathematics from grade one through junior college.

Thesis 1. *The school should guarantee functional*[3] *competence in mathematics to all who can possibly achieve it.*

Competence in mathematics in many respects parallels literacy in communication. The legal specification of literacy is fulfilled by the ability to write and to speak one's own language. In pioneer days, no doubt this ability was adequate. Today the U. S. Army uses the phrase "functional literacy" to imply fourth-grade ability. Though no one knows exactly what functional literacy means, except by arbitrary definition, it is clear that modern technology has stepped up the minimum requirements of literacy in communication.

In great-grandfather's day, when life was relatively simple, the ability to compute accurately when dealing with whole numbers, common fractions, decimals, and per cent was adequate for the ordinary affairs of life. There are good reasons for believing that the minimum requirement in mathematics for effective citizenship is moving upward and that it is already a big step higher than control of the four fundamental processes of arithmetic. Functional competence in mathematics in modern affairs seems almost as crucial as functional literacy in communication. Life in the armed forces today, for a boy who is merely a good computer, is decidedly limited—even the road to the rating as a sergeant is likely to be long and difficult if the candidate has not had some training in simple practical physics and the related mathematics. On the other hand, there is strenuous competition among the services to secure the boy who comes with this bit of mathematical training. Then, too, industry now provides a vast training program which usually includes mathematics in order to get what it needs for a great variety of jobs. The same increasing demand for higher mathematical competence is to be noted in changes in the nonvocational aspects of life today. Certain widely read magazines, the Sunday editions of newspapers of our larger cities, and even occasional radio programs all assume an understanding of certain basic mathematical ideas which are not fully understood by all of our citizens.

It is fair to ask, "Does anyone know what we mean by functional competence in mathematics, or is the phrase as vague as the word 'literacy'?" The answer is that the meaning has been fairly clear for over twenty years—the most authentic analysis having been given in the 1923 Report of the National Committee on Mathematical Requirements on *The Reorganization of Mathematics in Secondary Education*, which for the first time listed in considerable detail the specific mathematical objectives for citizenship. Moreover, we can now define functional competence in mathematics more concretely by utilizing the experiences of the armed forces. This source of information has already been investigated and the findings have been published in two committee reports.[4] These reports list basic concepts and useful skills that may safely be presumed to include the

[2] The Commission will at all times welcome critical reactions to its published reports. It is suggested that teachers of mathematics and mathematical clubs and associations examine the theses critically. Comments may be sent to any member of the Commission.

[3] Obviously the word *functional* here is used in the sense that the educationist uses the word, not in its mathematical sense.

[4] See the report of the Committee on "Pre-induction Courses in Mathematics," THE MATHEMATICS TEACHER, March, 1943. Pages 114–124; and the report of the committee on "Essential Mathematics for Minimum Army Needs," THE MATHEMATICS TEACHER, October, 1943. Pages 243–282.

mathematical materials of functional competence in mathematics for the armed services. But the reader will ask, Does the list of essentials in mathematics for minimum army needs apply to civilian life as well?

The mathematical phase of the training program in military schools, at least for the simpler tasks, came largely from industry. A glance at the list of approximately six hundred jobs in the army is sufficient to convince one that a very high percentage of these jobs have their counterparts in civilian life. Pending further investigation, it will be sensible to assume that the mathematics for minimum army needs, with only slight modification,[5] should be part of the general education of all our citizens. In the terminology of the educationist we can now outline with a good deal of assurance the mathematics of the *core curriculum*. Although the phrase "core curriculum" may disappear from pedagogical literature, there will persist the need for a common mathematical foundation for intelligent citizenship that will at the same time serve as a key to many "families of jobs." We need to make certain that the essentials for functional competence in mathematics are achieved by all who can learn them so that the proper guidance of the pupil may not be an impossible task.

The essentials for functional competence in mathematics are put as questions in the following Check List:

1. Can the pupil operate[6] effectively with whole numbers, common fractions, decimals, and per cents?

[5] It is not implied that the mathematics needed by the enlisted man and by the citizen in civilian life are *identical*. There are a few important differences. For example, as regards the mathematics of consumer education the enlisted man has in general little choice as to what he eats and what he wears, his budget problems are obviously not the same as one meets in civilian life.

[6] It is assumed that this item will be held to the needs stated in the two commission reports. For example, as regards operating with common fractions it is there suggested that the drill for mastery be limited to the denominators that people use in practical affairs.

2. Has he fixed the habit of estimating an answer before he does the computation and of verifying the answer afterward?

3. Does he have a clear understanding of ratio?

4. Is he skillful in the use of tables (including simple interpolation) as, for example: interest tables, tables of roots and powers, trigonometric functions, income tax tables, etc.?

5. Does he know how to use rounded numbers?

6. Can he find the square root of a number by table or by division?

7. Does he know the main guides that one should follow in collecting and interpreting data; can he use averages (mean, median, mode); can he make and interpret a graph (bar, line, circle, the graph of a formula, and of a linear equation)?

8. Does he have adequate ideas of point, line, angle, parallel lines, perpendicular lines, triangle (right, scalene, isosceles, and equilateral), parallelogram (including square and rectangle), trapezoid, circle, regular polygon, prism, cylinder, cone, and sphere?

9. Can he estimate, read, and construct an angle?

10. Can he use the Pythagorean relationship in a right triangle?

11. Can he with ruler and compasses construct a circle, a square, and a rectangle, transfer a line segment and an angle, bisect a line segment and an angle, copy a triangle, divide a line segment into more than two equal parts, draw a tangent to a circle, and draw a geometric figure to scale.

12. Does he know the meaning of a measurement, of a standard unit, of the largest possible error, of tolerance, and of the statement "a measurement is an approximation"?

13. Can he use certain measuring devices, such as an ordinary ruler, other rulers (graduated to thirty-seconds, to tenths of an inch, and to millimeters), compasses, protractor, graph paper, tape, calipers, micrometer?

14. Can he make a scale drawing and use a map intelligently—know the various forms employed in showing what scale is used—and is he able to find the distance between two points?

15. Does he understand the meaning of vector, and can he find the resultant of two forces?

16. Does he know how to use the most important metric units (meter, centimeter, millimeter, kilometer, gram, kilogram)?

17. In measuring length, area, volume, weight, time, temperature, angle, and speed, can he convert from one commonly used standard unit to another widely used standard unit; e.g., does he know the relation between yard and foot, inch and centimeter, etc.?

18. Can he use letters to represent numbers; i.e., does he understand the symbolism of algebra—does he know the meaning of exponent and coefficient?

19. Does he know the meaning of a formula—can he, for example, write an arithmetic rule as a formula, and can he substitute given values in order to find the value for a required unknown?

20. Does he understand signed numbers, and can he use them?

21. Does he understand what he is doing when he uses the axioms to change the form of a formula or when he finds the value of an unknown in a simple equation?

22. Does he know by memory certain widely used formulas relating to areas, volumes, and interest, and to distance, rate, and time?

23. Does he understand the meaning of similar triangles, and does he know how to use the fact that in similar triangles the ratios of corresponding sides are equal?

24. Can he, by means of a scale drawing, develop the meaning of tangent, sine, and cosine, and can he use a three- or four-place table of these ratios to solve a right triangle?

25. Can he solve simple verbal problems (in arithmetic, algebra, geometry, and trigonometry)?

26. Does he have the information useful in personal affairs, home, and community; e.g., planned spending, the argument for thrift, understanding necessary dealings with a bank, and keeping an expense account?

27. Is he mathematically conditioned for satisfactory adjustment to a first job in business; e.g., has he a start in understanding the keeping of a simple account, making change, and the arithmetic that illustrates the most common problems of communications, travel, and transportation?

28. Does he have a basis for dealing intelligently with the main problems of the consumer;[7] e.g., the cost of borrowing money, insurance to secure adequate protection against the numerous hazards of life, the wise management of money, and buying with a given income so as to get good values as regards both quantity and quality?

As has been suggested earlier, this thesis is a main goal for every teacher of mathematics throughout grades 1–14. Item No. 1 in the Check List is obviously the major part of the work of grades 1 to 6; the courses for grades 7 and 8 can and should make a substantial beginning in teaching most of the remaining items. General mathematics, commonly offered in the larger schools as an alternative to first year algebra in grade nine, should be constructed around these key concepts. The traditional sequential courses cannot be taught on the assumption that these basic matters have been learned well enough in earlier grades for effective use. Finally, it may well turn out that the subject matter suggested in the Check List is of more importance to many poorly prepared students in the junior college than some of the mathematical topics that they now are attempting to learn.

[7] For a more detailed and definite statement see *The Role of Mathematics in Consumer Education;* single copies may be secured without cost from the National Association of Secondary School Principals, 1201 16th St. N.W., Washington, D. C.

In our effort to insure functional competence in mathematics we need to keep the following in mind. (1) To an educated person the basic ideas are absurdly simple; typical examples are ratio, the nature of a measurement including degrees of accuracy, dependence, scale drawing, interpolation, the formula, tolerance, etc. (2) Although the basic ideas are simple, they are nevertheless very difficult for many persons to learn, and therefore require a long period of systematic teaching—instructors in the schools of the armed services, when they had to scrape the bottom of the barrel for human material, discovered that it took time and lots of it to teach a few simple mathematical ideas to *all* the boys of all the people. (3) While in general a good student who completes the traditional sequential courses will attain a mathematical maturity or power that exceeds the essentials, it is also true that a good student may complete the traditional courses (as they are now organized) without achieving functional competence; for example, there will almost certainly be gaps of the type revealed by tests in the armed services, and in any case, many students will not have as clear an understanding of some of the basic ideas as they should have. (4) A good student may become functionally competent in mathematics without taking any of the usual traditional sequential courses in high school, and he certainly can achieve the needed competence, as defined in the Check List, in less than four years, as for example, by taking one or two years of general mathematics. (5) Although the ability to compute effectively with whole numbers, common fractions, decimals, and per cents is still the crux of functional competence, this ability is achieved by only a fraction of the school's total population during the first eight grades, and is more likely to deteriorate than to improve in later grades unless something definite is done about it. (6) Although many of these ideas should be taught in earlier grades, this list of essentials for functional competence in mathematics should constitute the main goals of the work in grades 7 and 8. It is not to be inferred that all the essentials needed for functional competence in mathematics should be taught in grades seven and eight, nor that very many of these ideas can be completely taught to all pupils enrolled in these grades. However, grades 7 and 8 provide a golden opportunity for the mathematics teacher.

I. MATHEMATICS IN GRADES 1–6

If in the years to come arithmetic is to serve its proper function in the elementary curriculum, important changes need to be made both in the theory and in the practical procedures of instruction.

Thesis 2. *We must discard once for all the conception of arithmetic as a mere tool subject.*

For decades arithmetic has been classified as a tool subject.[8] Indeed, it is still so classified in many current courses of study. As a consequence, the tool conception of arithmetic has dominated instruction for a long time; indeed, for a time long enough to produce results by which the worth of that conception may be assessed. It is needless here to canvass all the evidence which has been accumulated. It must suffice to say that arithmetic taught purely as a tool subject has not achieved its purpose. It has failed to contribute its part in equipping our citizenry with the kind of mathematical competence which is required for effective and intelligent living in our culture.

The notion that arithmetic is nothing

[8] Judd, Chas. H., "The Fallacy of Treating School Subjects as 'Tool Subjects'." *Selected Topics in the Teaching of Mathematics*, Third Yearbook of the National Council of Teachers of Mathematics. Pages 1–10. Bureau of Publications, Teachers College, 525 West 120th Street, New York 27, N. Y. Price, $1.75 postpaid.

more than a tool subject is then condemned by its record in the schools. It can be condemned as well on theoretical grounds. It is said that arithmetic is just a tool subject because we never use it as an end in itself, but only as a means of dealing with the quantitative aspects of situations which are largely nonquantitative. By the same criterion history is also a tool subject and so is geography. We seldom think about history and geography just to be thinking about them. Rather, we employ them, as we employ arithmetic, in order to solve problems which transcend subject matter lines.

If it be pointed out that history and geography involve concepts, understandings, generalizations, and relationships, the same claim can be made for arithmetic. And by this same criterion arithmetic, like history and geography, becomes a content subject. As such, it has a body of knowledge which calls for the same kind of painstaking instruction that has always been accorded other traditional subjects.

It is possible, below, only to suggest the general character of this instruction. An account in greater detail appears in the published report of an earlier committee.[9]

Thesis 3. *We must conceive of arithmetic as having both a mathematical aim and a social aim.*

The fundamental reason for teaching arithmetic is represented in the social aim. No one can argue convincingly for an arithmetic which is sterile and functionless. If arithmetic does not contribute to more effective living, it has no place in the elementary curriculum. To achieve the social aim of arithmetic children must be led to see its worth and usefulness. In this connection "natural" as well as planned classroom and extra-classroom uses of number are especially serviceable. Children appreciate the value of arithmetic when it helps them to meet needs of vital importance to them. In meeting their needs through arithmetic they become sensitive to other possible uses, and they acquire the habit of using arithmetic as a perfectly normal way of adjusting to life situations. Children who have developed this sensitivity and this habit of use are well on the way to the attainment of the social aim of arithmetic.

We may grant the paramount importance of the social aim, and yet insist that it can be achieved only to a limited extent if the mathematical aim is neglected. The latter aim relates to the acquisition of the content of arithmetic, to the learning of arithmetical skills and ideas (concepts, principles, generalizations, and the like). Both skills and ideas should be made sensible to children through their mathematical relationships. This means that children must understand whole numbers, the number system, common fractions, decimal fractions, per cents, units of measure, etc.; that they must understand the functions of the basic operations, and that they must understand the rationale of our methods of computation. To teach well these understandings and the essential skills is to achieve the mathematical aim.

It is not a matter of having to choose between the mathematical aim and the social aim—we must realize both aims through our teaching. The next five theses are intended to show how this goal may be realized.

Thesis 4. *We must give more emphasis and much more careful attention to the development of meanings.*

In the discussion of the foregoing thesis "understand" is the key word. The purposes of arithmetic cannot be fully attained unless children understand what they learn and know when and how to use it. We face here the problem of developing meanings, both mathematical and social

[9] "Essential Mathematics for Minimum Army Needs." Report of a joint committee sponsored by the Civilian Pre-Induction Training Branch of the A. S. F. and by the U. S. Office of Education. THE MATHEMATICS TEACHER, XXXVI (October, 1943), 243–282.

meanings; and meanings have not customarily received their share of attention in classroom instruction.

Consider as a case in point the common practice of giving children rules instead of *developing* them; for example, of telling children where to write quotient figures in division, instead of helping them to see that their positions are predetermined by the principles of place value; or, of telling children to invert and multiply (a short cut) when they divide by a fraction, instead of developing, first, a rational explanation through the use of the common denominator.

Consider also how relatively empty of meaning are children's concepts of common and decimal fractions when in grades 5 and 6 they come to use them in abstract computation and in problem solving. There is small wonder that such children make absurd mistakes and still are utterly complacent about them. Nothing else should be expected. Not possessing the basic understandings, they can do little more than to acquire skills in a mechanical fashion. And when skills are acquired unintelligently, they can be used only unintelligently.

And, last of all, consider what is too frequently done in teaching the process of measurement and the units used in measuring. Traditionally children have been required to master various tables. This they have done with varying degrees of success, the more capable of them being able to recite the tables confidently and accurately. Yet, their learning may be exceedingly superficial, for they may have only the haziest ideas of the units involved (e.g., the length represented by a yard or the distance represented by a mile). And their learning may be without value, save in enabling them to make the abstract computations of classroom problems. Faced by practical situations, they may not be able to employ in an intelligent manner the units whose names they have memorized. Here, as elsewhere in arithmetic and in other subject matter areas, it is a mistake to accept glib verbalism as evidence of sound learning.

(a) Meanings do not just happen. Nor can they be imparted directly from teacher to pupils, as by having them memorize the language patterns in which meanings are couched. Instead, meanings grow out of experience, as that experience is analyzed and progressively re-organized in the thinking of the learner. In a word, each child creates his own meanings, accordingly teacher activities are perforce restricted to those of guidance. It is the function of the teacher to provide an abundance of relevant experiences and to assist the child to isolate the critical elements and to build them into the desired understandings.

(b) Meanings are not all-or-none affairs: they are relative matters. It is incorrect to say that a child either does or does not have an understanding. He may understand something well enough for one purpose but not well enough for another. The problem therefore is to help him extend and enrich his understandings to the needed limits.

(c) Experiences to develop meanings need to be arranged and ordered as carefully as are the experiences by which we develop computational skills. The first encounters with meanings should ordinarily occur in concrete situations of large personal significance to the learner. At first, and for some time, children should actually use the ruler to determine the length of objects and should find volumes by using cups and quart measures. In these activities their attention should be repeatedly called to the units which they are using, the purpose being to have them acquire an understanding of these units. Next they can be led to estimate lengths and capacities, the estimates being checked by actual measurement. When they can estimate with reasonable accuracy, it is time enough to have them move on to the next level of abstractness and to learn the relationships among inches, feet, and

yards, and among cups, quarts, and gallons, as these are summarized in the corresponding tables. As a matter of fact, having themselves (under guidance) discovered these relationships in connection with real measuring experiences, they may already have formulated the relationships and need only to complete their learning by organizing them into the standard tables.

What has been said about the development of meanings relating to measurement and units of measure is not limited, of course, to these ideas. It holds equally well for all the meanings commonly taught in arithmetic, or in any phase of mathematics, for that matter. It holds not only for mathematical meanings, but also for the social meanings of mathematics. In the latter connection it is imperative that teachers know how and when arithmetical skills and concepts are used, or may profitably be used, in the lives of children and of adults alike. In the absence of such knowledge it is improbable that children will have real experiences in employing the arithmetic they learn or that they will come to a rich appreciation of the significance of arithmetic in our culture.

Thesis 5. *We must abandon the idea that arithmetic can be taught incidentally or informally.*

It has been a popular notion in the last fifteen or twenty years that arithmetic can be satisfactorily taught in terms of pupils' "interests and needs" which result spontaneously from casual happenings or which are intentionally stimulated through planned units of work. The limitations of this instructional program have been pointed out many times.

(a) It is unlikely that children left to themselves will have enough number experiences or an adequate variety of experiences to develop a feeling of need for any but the simplest of arithmetical ideas and skills (e.g., counting). (b) Few teachers are sufficiently sensitive to the quantitative aspects of events to recognize them and to call attention to their presence in ordinary situations or to arrange for their presence in activity units. (c) When an effort is made to infuse arithmetic into activity units, not all children profit equally. Usually the more capable children "run away" with the project, and the less capable gain little if anything. (d) Number ideas and skills are not *learned* as such when they occur only as parts of larger experiences. True, these larger experiences may arouse a feeling of need for a new idea or skill, and so motivate learning; true, also, they may provide excellent opportunities to apply ideas and skills that have been already acquired. But they cannot produce or guarantee the learning. One learns little about the chemical nature of sea water by being immersed in it, or about the mechanical principles of a machine by operating it. (e) Mathematics, including arithmetic, has an inherent organization. This organization must be respected in learning. Teaching, to be effective, must be orderly and systematic; hence, arithmetic cannot be taught informally and incidentally.

Thesis 6. *We must realize that readiness for learning arithmetical ideas and skills is primarily the product of relevant experience, not the effect of merely becoming older.*

Of late, the so-called stepped-up curriculum has been rather generally accepted. Systematic instruction in arithmetic is deferred to Grade 3; the multiplication and division combinations are postponed a year or two, until Grade 4 or 5 or 6; the last computations with common fractions appear in Grade 6 or 7, instead of in Grade 5, and so on. The assumption is that children, by reason of their greater "maturity" in later grades, will easily learn ideas and skills which present undue difficulty when taught in the traditional grades.

(a) There is no magic in birthdays. So far as learning the school subjects is concerned, increase in age makes for readiness only (or at least predominantly) to the extent that extra time gives opportunity for relevant experience. Postponement of arithmetical topics can by itself be only

a questionable device for removing learning difficulties. (b) The earlier years are wasted. Children are deprived of ideas and skills which could give them surer control over their environment and their activities. (c) Not only this, but unless we are prepared to shift much of the traditional arithmetic of Grades 7 and 8 into the high school, there must inevitably be a jamming of content in Grades 3, 4, 5, and 6, with consequent superficiality in learning. (d) When children are adequately prepared (i.e., when they have been provided with relevant experiences), the traditional placement *for mastery* is not far from wrong. Admittedly, the more troublesome phases of topics may well be postponed; but a movement of the easier phases to earlier grades may be justified equally as well. In any case, the fundamental requirement is that children's number experiences be ordered so as best to assure progression in sound learning. (e) The research on readiness which is supposed to establish the need for postponement is exceedingly shaky, and does not warrant the conclusions which have been drawn. Certainly it does not justify the notion that the teacher's function is one of custodial care while waiting for the day of readiness to make learning possible. This idea is naïve in its psychological basis, and impracticable in operation.

Thesis 7. *We must learn to administer drill (repetitive practice) much more wisely.*

The pendulum swings from one extreme of excessive drill to the other extreme of no drill. It is possible to expect from drill both too much and too little. Drill—having children do essentially the same thing over and over again—cannot develop understanding. For this purpose varied experiences are called for. But once the desired degree of meaning has been generated, repetitive practice reduces the meaning to an easily managed thought pattern, it gives the learner confidence in what he does, and it protects the meaning against forgetting. In the case of skills it makes for efficiency of performance and leaves a basis for the building of later mastery. It may do even more than this. It may help to forestall the feelings of insecurity, frustration, and fear which too many children now experience in the arithmetic of the upper grades because of their lack of confident mastery.

From all this it follows that drill is to be prescribed, not in this or that grade, but at the critical time with respect to each separate idea and skill. This critical time comes in the last stage of learning, when earlier steps have been completed and when mastery is the goal.

Thesis 8. *We must evaluate learning in arithmetic more comprehensively than is common practice.*

Evaluation, like teaching, starts with a consideration of the outcomes, *all* the outcomes, which are to be achieved. In arithmetic these outcomes include more than skill in abstract computation and in problem solving. They include mathematical understandings, mathematical judgments, the ability to estimate and approximate, habits of use, and the like. If these outcomes are important for teaching, they are equally important from the standpoint of evaluation. And no program of evaluation which disregards these outcomes is adequately comprehensive.

Recognizing these facts, we shall use paper-and-pencil tests for what they are worth; but we must supplement these tests with other procedures, such as those of the interview, observation, the examination of work products, and the like. The spontaneous use of arithmetical ideas and skills in the ordinary happenings of the classroom and in planned activity units is, for example, the best kind of evidence that children are realizing the social aim of arithmetic.

II. THE MATHEMATICS OF GRADES SEVEN AND EIGHT

The theses which follow apply in essence to *all* seventh-grade and eighth-grade pupils of normal intelligence, ir-

respective of the type of school in which they are enrolled. The mathematical program of grades 7 and 8 should not be viewed as a problem by itself, but as an integral part of the total program as suggested in this Report.

Thesis 9. *The mathematical program of grades* **7** *and* **8** *should be essentially the same for all normal pupils.*

· This is not the place to begin differentiated courses. As was pointed out in the opening section of this Report, "we need to make certain that the essentials for functional competence are achieved by all who can learn them, in order that the proper guidance of the pupil may not be an impossible task." The seventh and eighth grades are crucial years in the attainment of that objective. Departure from these essentials will not only jeopardize the attainment of functional competence, but also result in large-scale retardation that is so difficult to correct later.

Regardless of the type of school or the community, the mathematical program of these grades should be designed to do three things:

(a) *Provide an adequate, organic continuation of the work of grades* **1–6**.

The seventh-grade teacher must, of course, begin the year's work with pupils "as they are." However, disabilities in arithmetic should not be regarded as an inevitable and permanent feature of the elementary school. On the contrary, the Commission wishes to assert that a fatalistic attitude toward this vexing issue is unwarranted, and that wherever large-scale weaknesses in arithmetic are observed, they should be corrected at the source, namely, in the elementary grades. On the basis of completely trustworthy evidence the claim is warranted that, under competent instruction, American children can and do acquire a satisfactory foundation in arithmetic in the elementary grades, and that the average child, when properly taught, enjoys arithmetic.

Disabilities in arithmetic should be located at the earliest possible moment with the aid of reliable inventory or diagnostic tests. An adequate remedial and maintenance program should be regarded as an integral part of the curriculum. Experience shows that shortages in arithmetic, if not promptly corrected, prevent or retard the pupil's progress in mathematics. In all the basic techniques of arithmetic the goal should be mastery. To attain this goal, extensive re-teaching is necessary in all cases of marked disability. Mere repetitive drill, devoid of real understanding, only aggravates the situation. A remedial program or a refresher course which is not anchored on insight, cannot be expected to produce lasting results. The secret of success in arithmetic is meaningful learning, combined with intelligent practice and functional application.

(b) *Provide a substantial beginning in achieving functional competence.*

It is the clear responsibility of mathematics to provide training that will make the pupil intelligent and efficient in dealing with the problems that he may meet in other school subjects, in the home and in his everyday reading and conversation. Therefore, as has been suggested earlier, many of the items in the Check List should be an important part of the program.

(c) *Provide a dependable foundation for subsequent courses in mathematics.*

Obviously, the mathematical program at any stage should not be determined exclusively by considerations of immediate needs and interests or of direct experience, but should also have regard for possible future needs.

Thesis 10. *The mathematics for grades* **7** *and* **8** *should be planned as a unified program and should be built around a few broad categories.*

Before it is possible to consider the grade placement of specific topics, it is necessary to develop a comprehensive plan for these two years. The program

should be organized[10] around: (1) number and computation; (2) the geometry of everyday life; (3) graphic representation; (4) an introduction to the essentials of elementary algebra (formula and equation).

It should be pointed out that these four categories are essential not merely for subsequent courses in mathematics, but also for the demands of modern industry, technology, and business, as well as for national defense.

Thesis 11. *The mathematics program of grades* **7** *and* **8** *should be so organized as to enable the pupils to achieve mathematical maturity and power.*

The teachers should constantly develop a genuine understanding and appreciation of the fundamental concepts, principles, and modes of thinking, and to develop real proficiency in using the basic techniques of mathematics.

The shortages in the mathematical training of young men entering the armed forces have in some quarters been attributed merely to "forgetting." However, there is convincing evidence that lack of understanding of the key concepts and principles, and of mathematical modes of thinking, is primarily to blame for widespread mathematical incompetence. The remedy is obvious. As Professor Dewey pointed out long ago, skills cannot be used effectively unless intelligence has played a part in their acquisition. Mechanical drill is not a substitute for understanding.

At all stages of mathematical instruction, the first concern of the teacher should be that of developing a real understanding of key concepts and principles. The pupil should know the underlying reasons for all the processes he is taught. In building for power the classroom work should involve more than mere theory. It calls for constant and insistent attention to significant applications. This is necessary for the sake both of effective motivation and of effective transfer. The range of possible and desirable mathematical applications is growing from year to year. As science and technology become increasingly important in the modern world, the functional uses of mathematics, likewise, become correspondingly urgent. In fact, throughout the recorded period of human history, mathematics has been a mirror of civilization. It seems destined to continue in that role.

III. MATHEMATICS IN GRADE NINE

Thesis 12. *The large*[11] *high school should provide in grade* **9** *a double track in mathematics, algebra for some and general mathematics for the rest.*

In the large high schools, the teacher of mathematics is confronted with a difficult problem, but, fortunately, it is one that can be solved. The range as regards both native ability and acquired competence of pupils is very great. For example, one commonly finds in the same class a few pupils with an I.Q. as low as 75 and several that have an I.Q. higher than 140. Many have not mastered the essentials whereas perhaps a fourth of the class are fully ready to undertake successfully the study of first-year algebra.

Far too often a school resorts to one or more of the following unsatisfactory solutions. (1) Only general mathematics is offered. This policy delays the beginning of the study of algebra for those who plan to take the long road to leadership in science and mathematics. (2) Only algebra is provided. This program inevitably results in a large number of frustrated pupils who fear and dislike mathematics, with a devastating lowering of standards in all algebra

[10] See, for example, The Report of the Joint Commission on "The Place of Mathematics in Secondary Education," The Fifteenth Yearbook of the National Council of Teachers of Mathematics, 1940, pp. 62 ff. The Bureau of Publications, Teachers College, 525 W. 120th St., New York, N. Y. Price, $1.75 postpaid.

[11] By a large high school the Commission means a school with more than 200 pupils. The special problem of the small high school is discussed in a separate section.

classes. In a survey by *Fortune*,[12] mathematics was the best-liked subject in the high school curriculum, but it also received a high vote as the least-liked subject. (3) Only a diluted algebra is taught. This practice is a shot that misses both targets. Algebra cannot fullfil its main purpose if the teacher resorts to a wide use of popular materials in the futile effort to meet the needs of pupils who should not be taking algebra at their stage of mathematical maturity. (4) Most of the pupils are encouraged to elect commercial arithmetic or consumer mathematics. This plan results in teaching much too early material that might be very worth while if taught several years later. In general, ninth grade pupils do not have the experiences that are needed for dealing with the mathematics of consumer problems in respectable fashion. It is difficult to motivate the arithmetic that may be needed on an unknown job that is still several years in the future. (5) The slowly maturing pupil is discouraged from taking any mathematics. This practice, as the experience of the schools of the armed forces has convincingly demonstrated, is an oversimplification of the problem. Too often this type of pupil leaves school with little control over the essentials needed later on a semitechnical job. (6) Many schools give drill work in arithmetic computation without stopping to teach the meaning of the processes used, without diagnosis, and even without motivation. This is merely giving a larger dose of a medicine that hasn't helped the pupil in earlier grades.

The situation in the ninth grade of the large high school clearly requires a double track in mathematics—algebra only for those whose ability and future outlook indicate to their advisers that they should take it and a good course in general mathematics for the rest. General mathematics for the ninth grade is here defined as a course that includes and emphasizes the elements of functional competence as outlined in the Check List on pages 197–198 of this report. It has been suggested earlier that the task of insuring functional competence cannot be completed for all pupils in the first eight grades. For many, this task must be continued at least through grade 9. The main purpose then of a general mathematics course in the ninth grade is to provide such experiences as will insure growth in understanding of the basic concepts and improvement in the necessary skills.

We come now to the phase of our problem that is difficult for many teachers of mathematics—the administration of general mathematics in a way that will make it respectable and desirable. Here the attitude of the teacher is the determining factor and far too often the teacher is, by training, disposed to propagandize unduly for algebra. As a matter of fact, algebra needs no propaganda—it will always be respected and it will always have great value for the person who should elect it. When, however, a teacher implies that the penalty for failure in algebra is transfer to general mathematics, an unwarranted halo of prestige is given to algebra that implies stigma and disrespect for general mathematics which should not be so if the general mathematics is properly organized and taught.

It should be made clear to pupils that the two parallel courses of the ninth grade are both tremendously worth while, but that they do have very different goals and experiences for pupils with different interests and needs. General mathematics is a more flexible course than algebra; it can more easily be adapted to different backgrounds and levels of ability. The material can and should be offered in such a way as to challenge the pupil to his best effort. Pupils should be told that general mathematics is *organized* differently, that it offers a greater *variety* of topics and that it is related *more directly* with immediate

[12] *Fortune*, November and December issues, 1942. Free reprints of this material may be secured by writing to the General Manager, *Fortune* Magazine, Time and Life Building, Rockefeller Center, New York, N. Y. See also THE MATHEMATICS TEACHER, February 1943. Page 83.

application. They may well be told that good work in general mathematics demands as much time and exertion as algebra. It is a fatal error to imply that in general mathematics anything will do. Shop teachers of the right kind certainly expect accuracy in computation and measurement beyond anything required in the ordinary academic class. General mathematics may seem an easier course than algebra, but that fact alone will not stigmatize the course in the opinion of a student body.

Classification should not be based on ability alone. In a strong mathematics department some of the best pupils in the ninth grade will be studying general mathematics. Differentiation to avoid trouble for either group should, as has been suggested, be based primarily on a difference of goals. A good argument for general mathematics is that a mastery of the essentials, as outlined in the Check List, will remove the feeling of insecurity from many ninth grade pupils who have never had the satisfaction of achievement. The best criterion, obviously, for selecting a pupil for the algebra class is the desire and the ability to do work of a high order of excellence. Therefore, unsatisfactory work in algebra should be tolerated for only a brief trial period—a semester would seem to be much too long. It should be noted that in guiding the pupil into the appropriate course, measures of reading, intelligence, and computation are also very helpful. Teachers should scrutinize critically all materials used in the guidance of pupils to make certain that they do not include statements that prejudice pupils against the general mathematics courses.

One or more general mathematics sections and at least one algebra section should be scheduled at the same hour in order to facilitate the transfer of pupils. The pupil who, at the end of a year of general mathematics, wishes to study algebra, may, if he has provided convincing evidence of adequate ability, be encouraged to elect the second semester of first-year algebra. Finally, it should be noted that the real hazard to general mathematics is the undesirable label unconsciously in the minds of many traditionally trained mathematics teachers. In schools where teachers of mathematics recognize the very great importance of general mathematics in the total offering, it readily becomes a popular course for many pupils. The road to algebra should be open to the pupil who matures slowly in mathematics, and general mathematics will be more highly regarded by pupils if it is made clear that it contributes to the successful pursuit of many things. The goal of a strong mathematics department should be to have every pupil in the appropriate course with no dissatisfied customer in any class.

Thesis 13. *In most schools first-year algebra should be evaluated in terms of good practice.*

Let no one assume that all is well with first-year algebra. There is a wide gap between first-year algebra as it is commonly taught and what good teachers everywhere have long demonstrated. The situation described in the following quotation[13] may still be found in many schools although it was written twenty-five years ago:

> The situation that needs to be met may best be illustrated by the case of algebra. Our elementary algebra is, in theory and symbolism, substantially what it was in the seventeenth century. The present standards of drill work, largely on non-essentials, were set up about fifty years ago. A considerable number of teachers, both in the secondary schools and the colleges, believe that the amount of time spent by pupils on abstract work in difficult problems in division, factoring, fractions, simultaneous equations, radicals, etc., is excessive; that such work leads to nothing important in the science, and adds but little to facility in the manipulation of algebraic forms.

[13] Quoted from a memorandum addressed to the General Education Board by a committee representing the Mathematical Association of America. This memorandum secured generous funds for the support of the National Committee on Mathematical Requirements. It is, therefore, of special interest to the student of mathematical education.

However, let us turn to the brighter side of the picture and see what is happening in the classrooms of competent teachers of first-year algebra in some parts of the country. It is likely that there are few, if any, high-school subjects that have been improved as much during the last quarter of a century as has first-year algebra.

The list of desirable trends is a long and impressive one. Today good teachers of algebra (1) reduce the manipulation of symbolism (nests of parentheses, four-story fractions, involved cases of factoring, difficult cases of simultaneous equations, etc.); (2) introduce a unit of from four to six weeks' duration on the trigonometry of the right triangle; (3) emphasize the notion of dependence or function; (4) teach with great care the meaning of a formula; (5) apply graphic techniques widely; (6) use the newer testing procedures for instructional purposes; (7) introduce symbolism gradually and through a variety of geometric and other illustrations; (8) discard the definitional approach and manage materials so that definitions as well as principles, processes and concepts stem from numerous and simple mathematical experiences; (9) make better provision for individual differences by providing problems of graded difficulty; (10) use a more sensible program of "drill" based on the fact that a pupil learns more quickly and remembers longer the things that he understands fully; (11) use a few simple, interesting, and practical applications to motivate each new principle and topic; (12) strive to improve the problem material by selecting functional applications (aviation, the school shop, general science, etc.); (13) make sagacious use of the bulletin board and other visual aids to enrich the subject; (14) utilize whenever possible laboratory or investigational techniques and seek to give the mathematics classroom the furniture, equipment, and atmosphere of a workroom; (15) know that reading ability fixes a low ceiling as to what can be achieved in problem solving in the case of many pupils; (16) recognize that it is far better to teach a few concepts well than to teach many concepts superficially; and (17) do what they can to restrict first-year algebra to those pupils who should study it and provide a course with sufficient rigor and continuity for subsequent courses.

IV. MATHEMATICS IN GRADES TEN TO TWELVE

The traditional sequential courses include the elements of algebra, plane geometry, solid geometry, and trigonometry. In some schools they may also include the elements of statistics, analytic geometry, and the Calculus. While an attempt is often made to correlate these various subjects, in general, they are taught separately. The very fact that the sequential courses are the oldest mathematics courses in the high school makes it difficult to change them. However, there is great opportunity for improvement. Again we might do well to follow the example of the industrialists and go forward with improved materials and more efficient methods.

Thesis 14. *The sequential courses should be reserved for those pupils who, having the requisite ability, desire or need such work.*

All the pupils in the high school need training in quantitative thinking, but it would be a mistake for this reason to require all of them to take the sequential courses in mathematics.

Pupils of ability should be informed that the sequential program in mathematics is a definite prerequisite for many lines of work.

The need for careful training in mathematics for those planning careers in the physical sciences, engineering, architecture, and similar fields is universally recognized. In order to give adequate preparation for these technical fields, the sequential courses must give attention to many phases of mathematics that are more detailed and abstract than is neces-

sary or desirable for most pupils. If the sequential courses are to fulfill the purpose for which they are intended, they cannot be emasculated to fit the needs of those of low ability and weak purpose.

Thesis 15. *Teachers of the traditional sequential courses must emphasize functional competence in mathematics.*

It has been assumed that pupils studying algebra, geometry, and the higher branches of mathematics of the high school not only retain their skills in arithmetic and other important topics of the junior high school but gain further understanding of these topics. This is not always the case. The war has taught us that success in the traditional sequential courses does not guarantee mastery of all the items in the Check List. For this reason the study of the fundamentals of arithmetic and of other phases of elementary mathematics taught in grades 7 and 8 must be continued in the senior school.

Most of these topics can be fitted into the regular sequential work and become an integral part of it. For example, instead of being content with the understanding of principles in algebra and geometry and applications of these principles in numerical exercises with small whole numbers as is often done, good practice in arithmetic can be obtained by using larger numbers, fractions, mixed numbers, and decimals.

It follows that provision should be made for periodical checks on pupils' understanding of these topics. Topics not clearly understood should be retaught and adequate practice provided for those students who need it.

Thesis 16. *The main objective of the sequential courses should be to develop mathematical power.*

Drill on the manipulation of mathematical symbols not accompanied by clear understanding of the underlying concepts and principles is of little value. When such drill is discontinued, the ability is soon lost. Power is attained when the learner understands the relationships involved well enough to apply them in new and varied situations.

Thesis 17. *The work of each year should be organized into a few large units built around key concepts and fundamental principles.*

In a sequential course the major emphasis should be on concepts and principles. It has long been urged that some of the more complex manipulations in algebra be treated lightly or omitted altogether, provided the basic ideas are mastered. For future work in mathematics and the sciences, the basic ideas are more important than complex details. The student who knows the fundamental meanings and has complete mastery over the simpler manipulations in connection with the basic ideas can gain the complex details when and if necessary. For example, in a first course in algebra, one of the large units may well be "algebra as generalized arithmetic." The unit devised on this idea would point two ways, backward to arithmetic already learned, and forward to more advanced algebra. It would carry with it the symbolism of algebra, would bring together and clarify many of the relationships of arithmetic, and give meaning to a host of isolated topics usually considered as merely formal algebra.

Then too, we should not continue to give so much time to continuous logical development in geometry. Once a student has learned what it means to prove a statement deductively, it is not necessary to devote an entire year to deductive proof. Of course, we should continue proofs throughout the year, and in subsequent years, just as we should continue arithmetic. Meanings and skills are not established once for all. Some of the theorems can be postulated after careful laboratory work, while others can be discussed informally.

In solid geometry proofs might be restricted to those theorems which deal with lines and planes in space, and with the geometry of the sphere. The metric prop-

erties of solids, including the sphere, may well be developed informally.

A common complaint of teachers is that there are so many topics to teach in any one year that they cannot "finish the book." If we teach in terms of the mastery of key concepts and fundamental ideas, the pages not covered will not matter so much.

While there is considerable continuity in the sequential courses, there is not as much as has been generally believed. As taught, the various topics in algebra and geometry seem, to the pupil at least, to be quite unrelated. The continuity can be seen better in terms of large ideas rather than in terms of details.

Planning a year's work in terms of large units built around key concepts and fundamental principles will also help to solve the problems of integration of the various subjects. It is obvious that natural interrelations between algebra, geometry, and trigonometry should be emphasized.

Thesis 18. *Simple and sensible applications to many fields must appear much more frequently in the sequential courses than they have in the past.*

Applications of mathematics to problems of industry, physical science, aviation and business should be used for purposes of motivation, illustration, and transfer. Mathematics teachers must become sufficiently familiar with these fields so that they can choose the applications wisely. But mathematics cannot be taught solely through its applications. A few simple applications at every advanced step should be used for motivation and as a means of increasing the possibility of transfer. They cannot be an end in themselves.

Thesis 19. *New and better courses should be provided in the high schools for a large fraction of the school's population whose mathematical needs are not well met in the traditional sequential courses.*

As everyone knows, there are pupils—very large groups—who do not elect the sequential courses, and many of whom are maladjusted in case they do. Such pupils as these will never be satisfied with a purely academic program. Many of them have the ability to render valuable services with a bit of mathematical training as is evidenced by the several hundred thousand boys who held important semi-technical jobs in the military activities as a result of mathematics they studied *after* they left the regular schools. The military training program took these boys and taught them a simple technical science and the related mathematics. We must give more attention to the needs of industry, which the traditional mathematics teacher has neglected far too long. We must provide a more realistic curriculum for the large number of persons who will continue to be absorbed fairly early in life by industry, trade, farm, and business. Then, too, we must provide a course that will give them greater mathematical security in practical affairs, such as budgets, insurance, taxation, and the like.

Since there are in our high schools these large groups of pupils whose needs cannot possibly be met by traditional mathematics courses, the sensible thing to do is to provide good courses with very different goals and experiences for groups with different needs. Furthermore, as has been suggested earlier, we must somehow do this in a manner that does not offend any group.

What kind of courses are suggested by these varied needs? It is obvious that a year or even two years of mathematics paralleling the sequential courses, in grades 10 and 11, will serve a useful purpose for a very large part of the total school population. The content of this mathematics would clearly embrace substantial materials from at least several of the following areas: mathematics as related to trades and shop work; commerce and business; industry; agriculture. It is also clear that every pupil is potentially both citizen and consumer; hence all pupils should be given some understanding of the persistent prob-

lems that confront most of our families; viz., social security, taxation, insurance against the numerous hazards of life, and ways and means of stretching the dollar in order to buy the maximum of material comforts and values with a given income.

To be sure, this does not preclude the possibility that certain schools in certain areas will deem it advisable to treat these separate areas more extensively and specifically through specialized courses.

In the early months of the war, a refresher course in mathematics was popular among seniors in the high schools. Even though the term "refresher" may disappear, there will probably always be pupils in the late years of the senior high school who have a feeling of insecurity with respect to mathematics. It would seem that there is a place, at least in the large high school, for a course that reteaches the essentials of mathematics with new and fresh materials consisting of practical applications that are simple and interesting.

Then too there will probably always be some pupils who arrive at the eleventh or twelfth grades without having taken any mathematics beyond the eighth grade. These pupils may wish to rectify a mistake, and so will want a course that will make them as competent as possible in a relatively short period of time. The Commission believes that, so long as this need exists, the larger schools, at least, should continue to provide a course for pupils in the eleventh and twelfth grades with a content for the most part as outlined in a report entitled *Preinduction Courses in Mathematics*.[14]

There is here no implication that the traditional sequential courses will be less important in the future than they have been in the past. Since the turn of the century the high school has been facing a double responsibility. It must train for leadership, and it must provide a broad education in terms of effective citizenship, in the home, the community, the state and the world. It is not a question as to whether special attention shall be given to either group; *both jobs must be done*. However, it is the main thesis of the Commission that there are at present large neglected groups of pupils whose needs cannot possibly be met in traditional courses, and for whom new and better courses should now be provided.

Thesis 20. *The small high school can and should provide a better program in mathematics.*

Many persons do not realize that more than two-thirds of all high schools are small, with certainly fewer than 200 students and probably fewer than 8 teachers. Such small high schools enroll in all more than a million pupils. Professional literature dealing with the mathematics of the small high school is meager. For example, the important committee reports on mathematical education devote very little space to this problem. In amazing fashion a considerable fraction of the population of our high schools has been overlooked altogether in formulating programs for the betterment of mathematics.

The mathematical offering of the small school is necessarily limited by the following conditions: (1) the rate of turnover of teachers is high; in a given year, most, indeed all, of the teachers may be new to their positions; (2) the teachers have little experience; in a three-teacher school, all may be beginners; (3) the member of the staff teaching mathematics may have had little or no training in the field; (4) the cost of instruction per pupil is high; (5) the library and the storage space within the classrooms for supplementary materials are inadequate; (6) there is seldom a classroom devoted exclusively to the teaching of mathematics, and (7) the teacher of mathematics may have to teach in two or more other fields.

As a result, the curriculum in mathematics is meager and remote from the pupils' needs. Thus, a small high school with five or six pupils in a class may offer only two

[14] See THE MATHEMATICS TEACHER for March 1943.

years of mathematics. These two years may be organized according to any one of many patterns that are in most cases unrealistic when evaluated in terms of the pupils' mathematical needs. For example, a school may offer a year of formal algebra and a year of demonstrative geometry. Presumably the aim in this case is to prepare all pupils for college even though the school may have sent relatively few of its graduates to college in its entire history. At the other extreme a school may offer only one year of commercial arithmetic and a year of agricultural mathematics with little emphasis on basic concepts and fundamental principles. Then too, many schools offer only a year of general mathematics and a year of commercial arithmetic. In such schools the occasional pupil who should be preparing himself for future leadership in science and mathematics, graduates with far less of the sequential mathematics than he should have.

What constructive suggestions can be made for the improvement of mathematics for the small school? Let us assume that we are dealing with a school that has only six pupils in each grade from 9 to 12 inclusive. Let us further assume that we can use only as much of one teacher's time as is represented by one-third of the school day (two periods of an hour each). For this small school the Commission makes the following suggestions.

(a) *Offer two courses simultaneously within the same class period.* Instead of general mathematics or algebra, one or the other, it is proposed that both be taught. The teacher of the one-room rural school has always taught from six to eight groups simultaneously; it seems reasonable that a high school teacher can teach two. In fact, an experiment in Indiana has demonstrated the feasibility of such a procedure. In our hypothetical school, one, or perhaps two, of the six ninth-grade pupils might, under proper guidance, elect algebra and the remainder study general mathematics. A generous fraction of each hour period should be a work-period with an up-to-date textbook. The algebra pupils might not need more than thirty minutes per week for the checking and guidance of their work. There are instances on record where pupils have done a year's work in algebra to prepare themselves for rigorous college entrance examinations without utilizing more than a dozen hours of a teacher's or tutor's time. The notion that a pupil must recite for five hours a week is outmoded.

Our hypothetical school might, in the tenth year, offer both geometry and a course in consumer mathematics to which pupils in the tenth, eleventh, and twelfth grades might be admitted. Thus the mathematics offering of a small high school might well be broadened and made more flexible. If a school can afford instruction in mathematics for three or four periods of the school day, its offerings can, by this plan, be as wide as is now commonly found in much larger high schools.

(b) *Provide correspondence courses in the small high schools.* The small school can extend its mathematics offering by encouraging interested and capable pupils to elect correspondence courses. A great variety of correspondence courses are now provided by commercial organizations. Moreover, it seems reasonable to assume that the many correspondence courses now available to the men and women of the armed services may be made available also to public education in the postwar period. It is gratifying to note that at least one state (Wisconsin) has authorized school boards to buy correspondence courses, and in other states there seem to be no legal obstacles in the way of a school board that desires to pay for correspondence courses. The Wisconsin law reads as follows:

> The board of any school district which operates a high school may contract with the university extension division of the University of Wisconsin for extension courses for pupils enrolled in such high schools. The cost of such contract shall be paid out of school district funds and shall be included in the cost of operation and maintenance of the high school districts which enter into such contract for the purpose of computing tuition costs.

During the war, over a million men have taken correspondence courses under conditions that were often very difficult. It would seem that pupils in the small high school would have a much better chance to succeed in correspondence courses than men in the armed forces for the reason that the local mathematics teacher might service such work in the courses that he happens to be teaching. Under this plan a pupil who wishes to study first year algebra might attend the class in general mathematics or in geometry, and take a correspondence course in algebra under the general supervision of the teacher. Since the teacher of our hypothetical school has only a few regular students, the implementation of a correspondence course for one or two pupils would obviously not be an impossible task.

(c) *Increase the number of courses by cycling.* Our hypothetical school might well offer algebra and general mathematics one year, geometry and general mathematics the next year, and general mathematics and another course consisting of a third semester of algebra and a semester of trigonometry in the following year. Under adequate guidance and by careful planning early in his high school career a pupil can thus get at least two years of general mathematics or even three years of sequential mathematics by the time he graduates from high school. It is assumed that some schools might wish to substitute a course in related mathematics, commercial arithmetic, consumer mathematics, and the like, for any one of the courses used in the preceding illustrations. However, in cycling, careful planning and adequate guidance are necessary in order to avoid a situation in which a prerequisite course has been offered in the wrong year for a group of students.

From the foregoing, it becomes obvious that the offerings of the small high school do not need to be as limited as they so often are. Incidentally, by this plan classes will be approximately doubled in enrollment with the cost of mathematical instruction per pupil sharply reduced. Finally, it is suggested that the superintendent and the board of education employ at least one teacher with a respectable minor or major in mathematics in order that the extended offering may be properly taught.

V. MATHEMATICS IN THE JUNIOR COLLEGE

It is now rather generally agreed that one of the main functions of the junior college is to serve as somewhat of a "community institute" providing educational opportunities, which otherwise might be inaccessible, to a large number of educable youth. The curriculum of each junior college, therefore, becomes something of a local enterprise in that it needs to be organized and administered in relation to the pattern of living in its community. This statement does not restrict the curriculum within the narrow limits for living always in that community; rather it calls attention to the perspective needed for a really functional educational program.

From the point of view of interest in mathematics, the student body of the junior college will divide itself into three major groups: *Group I*—those students who desire some knowledge of mathematics merely as a part of their cultural background; *Group II*—those students who need a minimum of certain mathematical prerequisites because of their desire to follow specific vocational interests; *Group III*—those students who have major mathematical needs because they plan a career in some field such as engineering, natural science, or pure mathematics.

Thesis 21. *The junior college should offer at least one year of mathematics which is general in appeal, flexible in purpose, challenging in content, and functional in service.*

In any junior college there is likely to be a group of students who feel that they do not care to take a traditional course in mathematics (Group I). They are not interested in intricate calculations and ex-

cessive manipulation. Many of them do not have, at least they think they do not have, the special aptitude necessary for understanding such work. It does not follow, however, that these students would elect to by-pass all mathematics courses. Many of them might find interesting challenge in courses containing those mathematical ideas and experiences which are an essential part of a liberal education. A basic general course in mathematics might offer these individuals opportunities for development which the traditional courses in freshman mathematics do not offer.

A course such as the one suggested above could also have definite cultural content. This would be particularly true if the mathematical concepts and techniques developed were presented as an important part of the history of thought, and were interpreted in terms of social usefulness. Such a presentation of materials would enhance the opportunities for richer significance, deeper appreciation, and clearer understanding of all mathematical concepts and techniques. Those students who have an interest in mathematics only as a part of their liberal education would learn from such a course something of the vital significance of mathematics as an integral element of the cultures of the world.

Thesis 22. *The junior college program should provide for a one-year pre-vocational course in mathematics.*

There is a wide range of vocations which require that mathematics in varying amounts be given a significant place in the program of preliminary training, and the number is likely to increase in the post-war period. It, therefore, follows that the students of Group II will very likely be one of the largest and most important groups in any junior college program. The mathematical requirements of many of these vocations will be such that they can be satisfied by a one-year program in basic mathematical concepts and techniques. It is the responsibility of teachers of mathematics to determine this body of basic mathematics and to organize it into a general one-year program which will take care of the pre-professional requirements which this group of students will need to meet. For example, there are many prospective teachers of other school subjects who are not prepared for, and who, therefore, cannot be expected to become interested in conventional courses in college algebra and trigonometry. There is, however, an abundant reservoir of mathematical information of rich cultural value and essential educational significance to all who aspire to teach in our nation's schools. Similarly, there are many students preparing for pharmacy, medicine, business, agriculture, industry, and many other fields of service, who are forced to elect traditional freshman mathematics instead of being challenged by a more functional program of mathematical training.

There is a great need for a basic one-year program in mathematics at the junior college level such as has been outlined by the Joint Commission.[15] Needless to say, in a movement so young as the junior college, considerable experimentation will be necessary to determine the desirable content of such a course. The function of such a course will indeed vary from time to time, and from one community to another.

Thesis 23. *The junior college program should make ample provision for the student with a major interest in mathematics.*

One of the imporant functions of the junior college is that of preparing its students for subsequent work at a more advanced level of instruction. The third group of students, namely, those who have a major interest in mathematics, should be provided with the opportunity for becoming more proficient in fundamental mathematical techniques and for broadening their contacts with basic mathematical concepts and skills. Their horizon of mathematical needs will vary considerably

[15] Report of the Joint Commission, *op. cit.*, p. 159.

from the more modest demands of the less technical phases of biology to the maximum prescriptions of a major in mathematics. The program for this group of students should be characterized by a clear current of challenging mathematical thought and meaningful practice.

A program allowing for such flexibility of mathematical training as is suggested in this report will greatly enhance the opportunities of the junior college for effective educational service. It should be of particular significance in the immediate post-war period, when the varying demands of the returning war veterans will tend to make the planning and administering of educational programs very difficult.

VI. THE EDUCATION OF TEACHERS OF MATHEMATICS

A. In Grades 1–8

Most mathematics courses in the elementary school are taught by teachers who have responsibilities with respect to other bodies of subject matter. On this account the proposals to be advanced below are offered with full recognition of the necessity to prepare teachers for duties in addition to those relating to arithmetic (elementary mathematics).

Thesis 24. *All students who are likely to teach mathematics in Grades 1–8 should, as a minimum, demonstrate competence over the whole range of subject matter which may be taught in these grades.*

It is a mistake to assume that teachers need to know only the subject matter which they will teach. In the first place, they should be expected to have more mathematical competence than we should insist upon for the average adult. In the second place, they cannot orient their instruction properly if they do not recognize the potential consequences of their instruction for successful accomplishment on the part of their pupils in later grades. For example, primary-grade teachers may see little sense in teaching the nature of our number system if they do not understand how this knowledge facilitates the meaningful learning of computation in Grades 3 to 6.

Each prospective teacher should be expected to achieve and to demonstrate mathematical competence. Teachers of Grades 1 to 6 should do this on their own responsibility, without course credit for whatever special study they may need to make. Instead, with or without coaching, such students should prepare themselves until they can make a satisfactory score on an acceptable examination. It is suggested that this score might be equivalent to the tenth grade norms in computation and problem solving on some comprehensive and reliable standard test. For teachers of Grades 7 and 8 this criterion is wholly inadequate, for competence needs to be assured over a much wider range of subject matter, to include something beyond the elements of algebra, geometry, and trigonometry which are involved in the courses for Grades 7 and 8.

Thesis 25. *Teachers of mathematics in Grades 1–8 should have special course work relating to subject matter as well as to the teaching process, as detailed below.*

Thesis 24 above merely assures mathematical competence as such competence has traditionally been conceived. An earlier section of this report made clear the shortcomings of this conception, at least as far as arithmetic is concerned. It therefore becomes necessary to outline other aspects of mathematical preparation, now too commonly neglected, which are nevertheless indispensable parts of the equipment for effective teaching. As will be evident, the provision of this professional equipment places heavy responsibilities upon instructors in the college courses which are called for. This is particularly true in the case of courses for teachers in Grades 7 and 8. In such courses the instructors might well be mathematicians who, besides their interest in the improvement of instruction, have rich backgrounds of experience in the social, vocational, and

industrial uses of mathematics.

(*a*) *Theory and background of elementary mathematics.*—To live effectively and intelligently in our culture we must of course be able to compute and to solve verbal problems quickly, accurately, and confidently. But we must also be equipped with meanings, generalizations, appreciations of relationships, and the like. In a word, we need to know something of the theory and background of elementary mathematics.

Important as this knowledge is for the average citizen, it is vastly more important for teachers. Lacking it, teachers will scarcely be qualified to help their pupils to acquire it. Hence, at some point in their education, prospective teachers of Grades 1 to 6 need to learn a good deal about the nature of our decimal number system and the story of its development, the evolution of fractional notation, the theory and history of measurement, the functions of the fundamental operations with whole numbers, fractions, and decimals, and the rationale and history of computation. Teachers of Grades 7 and 8 need all this, but they obviously need much more than this. Their course work in algebra, geometry, and trigonometry should be correspondingly extended to include the social and historical background of these more advanced mathematical subjects.

(*b*) *Important applications.*—The foregoing paragraphs are intended to suggest the type of preparation teachers should have in order to realize as fully as possible the mathematical aim of their subject matter. But steps need also to be taken to assure equivalent preparation in order that they may realize the social aim as well. It is possible to know mathematics as a closed system of ideas and skills without being able to apply it.

Teachers of Grades 1–6 who know arithmetic in this narrow way are insensitive to the contribution of arithmetic to social progress and to sound and effective individual adjustment in our present culture. The course in the teaching of arithmetic should remove this deficiency. It can do so if the topics mentioned in the preceding section are treated with respect to their social implications and if students are encouraged to note systematically the applications of mathematical concepts and processes to everyday life. The same generalization applies in the case of teachers in Grades 7 and 8, of course with due recognition of the expanded area of their mathematical subject matter. Beyond the material which they will teach they need backgrounds in the social and economic uses of mathematics. Thus, to be able to teach with confidence and assurance such topics as buying a home, the cost of running an automobile, making provision for the future (savings, insurance, social security, and investments), protection against large losses (fire and liability insurance), the prospective teacher needs course work and real experiences in these economic applications of mathematics.

(*c*) *Supplementary instructional equipment.*—Prospective teachers should have access to new textbook and workbook series, to some of the better courses of study, to standard tests, and to the newer devices and aids for teaching, such as models, films, film strips, slides, and the like. They should have access to a reasonable amount of this equipment; but, more than this, they should have opportunity to examine it critically and to note special advantages and limitations. No single textbook or workbook series, for example, has a monopoly on good systems of organization, on superior methods of meeting individual differences, on ingenious testing devices, and on commendable developmental explanations. And teachers should not expect to find all that is good and all that is important in the series which they happen to teach. Moreover, they should know and understand the value of the many excellent multi-sensory aids which are being made available in increasing numbers. (See Section VII of this report.)

(*d*) *Methods of teaching.*—In courses in the teaching of mathematics, the methods

which are most commonly described and exemplified are those which relate to facts and to mechanical skills. With new emphasis on the mathematical aim of arithmetic, prospective teachers must learn how to develop meanings, understandings, generalizations, a sure grasp of relationships, and the like. And with the new emphasis on the social aim, they must know how to engender in their pupils sensitivity to the usefulness of number and of measurement in life.

(e) *Student teaching.*—Provision should be made for adequate experience in student teaching. In considering this need, see Thesis 32.

(f) *Procedures for comprehensive evaluation.*—It was stated above that prospective teachers should know about the better standard tests in mathematics. But, good as they are for some purposes, these instruments are inadequate for comprehensive evaluation. Almost without exception they measure skill in computation and ability in problem solving and these outcomes only. If the other outcomes which are inherent in the mathematical and social aims are to be evaluated at all, they must be evaluated by means other than published tests.

For comprehensive evaluation prospective teachers need, first of all, to have a clear conception of the purposes of mathematics. They need, second, to know and to be able to identify the kinds of behavior which will be exhibited both by their pupils who are achieving these purposes and by those who are not. Third, they must have considerable ingenuity in devising situations for testing, observing, and interviewing, in order to elicit critical types of behavior. And, fourth, they must have confidence in their judgment, however subjective it may be, as a basis for evaluating progress in learning.

(g) *Research literature.*—In the end, it will be through competent research that we shall arrive at greatly improved teaching procedures. There are at present more than two thousand published reports of quantitative investigations, some good, some bad, on the teaching of mathematics. Yet, many teachers are unaware of this large body of information, and still fewer make use of it. This condition should be remedied.

Only a few teachers will become producers of research, but all of them can and should become intelligent consumers of research. This latter purpose can be achieved if at some time in their education they can be brought into contact with a few selected research reports. These reports they should study under guidance, with the view to developing habits of critical analysis. When they are possessed of these habits, they can profit immediately from the findings of new studies rather than have to wait, as is now the case, for these findings to work their way slowly into textbooks and courses of study. In this connection the Yearbooks of The National Council of Teachers of Mathematics should be helpful.

B. In Grades 9–12

The *optimum* training of mathematics teachers for grades 9–12 should be based on the functions they should be able to perform and on the objectives they should seek for their pupils. A well-prepared teacher of mathematics should have adequate training so that he can meet all classroom situations with that assurance which can be based only on wide knowledge and rich background. The following theses, based on this optimum training, suggest course work and study in mathematics, education, and related fields over a period of five years.

Thesis 26. *The teacher of mathematics should have a wide background in the subjects he will be called upon to teach.*

It will not be possible to accomplish this program completely with most prospective teachers during their undergraduate training. However, enough training may be given to enable the teacher to get well on the way to complete accomplish-

ment. This thesis means that the teacher's study of mathematics must include courses in college mathematics and not be merely a review of high school mathematics (algebra, plane and solid geometry, and trigonometry).

A teacher cannot teach mathematics with competence unless he knows enough about the subject to understand what elements are important for pupils in their possible life needs or future study. He cannot intelligently aid in reorganizing the high school curriculum in mathematics without an adequate reserve of mathematical knowledge. Without this kind of training he will not see that pupils need basic understandings as well as manipulative skill if they are to develop power in dealing with quantitative situations.

The mathematical background should include work in trigonometry and solid geometry if these have not been studied in high school. It should also include analytic geometry and calculus (elements of these subjects are taught in many high schools) and a course in college geometry beyond the secondary course in synthetic geometry. Also advisable are a course in the theory of equations and a course in spherical trigonometry with applications to global geometry, astronomy, and mapping; so also is a course in the history of mathematics, with emphasis on the historical development of computation and of elementary mathematics.

Some knowledge of the foundations of mathematics is indispensable for a well-trained teacher of mathematics. It is not likely that students will get it outside of class work. It may easily be included in the courses in college algebra and geometry.

At some time in his training the teacher should learn the use of, and the elementary problems of, the transit, sextant, slide rule, other mechanical computers, and related elementary problems. Not to be neglected is the habit of browsing in the college library that gives a knowledge of those recreational topics so useful in conducting mathematics clubs. College mathematics clubs, and the sponsoring of high school clubs during apprentice teaching will help greatly.

Other courses from which selections may be made and which will widen the teacher's background are: elementary statistics and educational measurements, the elements of non-euclidean geometry, projective or descriptive geometry, and the mathematics of finance. Selections should not, however, be made from these courses at the expense of the other training outlined above in this section.

Thesis 27. *The mathematics teacher should have a sound background in related fields.*

Courses in physics, mechanics, astronomy, navigation, economics, business problems, and the like add much to the teacher's ability to draw upon other fields and to understand and use vital applications from these fields in his teaching.

Thesis 28. *The mathematics teacher should have adequate training in the teaching of mathematics, including arithmetic.*

The background developed in previous paragraphs is of little use to the teacher unless he can teach with skill. Therefore he should take courses in the history and philosophy of education, psychology, and the techniques and problems of teaching in general, as well as specific methods in one or more mathematics subjects.

The mathematics teacher should be thoroughly familiar with the methods of teaching arithmetic. The war has shown a need for emphasis on arithmetic by high school teachers. There is a rather widespread belief among high school teachers that a knowledge of higher mathematics carries with it skill in teaching arithmetic. Such is not the case. A teacher cannot adequately remedy earlier defects in arithmetic teaching without a knowledge of how arithmetic should be taught, nor can he, without such knowledge, devise adequate diagnostic tests or conduct remedial work in arithmetic. The need is for specific

training in arithmetic for prospective high-school teachers.

As a part of his training in teaching methods he should become acquainted with those multi-sensory aids that are available through commercial sources, motion pictures, film strips, and models. In time he should become adept in devising and making for his own class use simple models, devices, and even film strips, to aid him in teaching.

Thesis 29. *The courses in mathematical subject matter for the prospective mathematics teacher should be professionalized.*

Since the objectives to be sought in training a high school teacher are obviously not the same as the objectives sought in training a research scholar or an engineer, college instructors in mathematics should be closely connected with the teaching of mathematics in secondary schools, should have an intimate knowledge of the problems that teachers in such schools have to meet, and should be able to tie in the college courses with problems in secondary teaching.

Thesis 30. *It is desirable that a mathematics teacher acquire a background of experience in practical fields where mathematics is used.*

He should have opportunity for experience in such fields as the general shop, machine shop, the making and reading of simple blue prints, and surveying. A few colleges and universities are providing courses for prospective teachers of mathematics which include a semester of experience in holding a job in an industry or a business. It would seem feasible and certainly desirable for a teacher of mathematics to devote at least a summer or two in learning a variety of jobs in one of the large manufacturing plants. All such experiences provide illustrations of mathematical uses.

Thesis 31. *The minimum training for mathematics teachers in small high schools should be a college minor in mathematics.*

This report would not be complete without mention of teachers in the smaller high schools who must teach one or more subjects besides mathematics and who may have a major interest in some subject other than mathematics. Under these circumstances the training outlined for mathematics teachers in larger high schools, whose major interest is in mathematics, cannot be expected or obtained. In suggesting a minimum training for such teachers there is always the chance that the minimum may be misinterpreted as a satisfactory standard. This error must not be made. If these teachers continue to teach mathematics, they should plan to take undergraduate or graduate work to bring their background and training up to the standard suggested in these theses.

Since the training outlined in this thesis must be considered as an absolute minimum, the state should insist that, if the teacher is to continue teaching mathematics, he must improve his preparation. This improvement should follow the suggestions made earlier in this section.

Thesis 32. *Provision should be made for the continuous education of teachers in service.*

Prospective teachers for all grades should have extensive opportunities to engage in student teaching under skillful supervision and guidance. To say that one learns to teach by teaching is not to disparage the more formal and academic aspects of teacher education. It is simply to recognize that such instruction gains greatly in meaning and effectiveness when it is translated into the activities of concrete teaching experience. Ideally, student teaching should be started in a campus or laboratory school where good teaching and competent supervision is, in general, rather easily provided. It should be supplemented by experience in situations involving run-of-the-mill pupils, a feeling of full responsibility for the results produced, and at least reasonably good supervision.

A teacher should serve an internship

which will enable him to observe good teaching and, under the guidance of a critic teacher, allow him to practice what he has observed. A training school, no matter how fine, can not provide all the variety of experiences necessary. A student teacher needs an internship under competent supervision in a typical school situation with about half a teaching load and with enough pay so that he can concentrate on becoming a good teacher.

VII. MULTISENSORY AIDS IN MATHEMATICS

Thesis 33. *Mathematics teachers need to give careful consideration to the possibilities of multi-sensory aids.*[16]

The schools of the armed forces have made extensive use of training aids which may be listed as, (1) motion pictures, (2) film strips and slides, (3) graphic charts and pictures, (4) models and other equipment, and (5) recordings. The figures for production and use are fantastic.[17] As of January, 1944, 2200 film-strip subjects had been produced by the Training Film Section of the Navy alone! Figures on the use of other visual aids, charts, graphs, models, recordings, etc., would likewise exceed the imagination.

Training aids are useful in a great variety of testing and learning situations. Training aids are being used in the armed forces in about every type of training and in about every conceivable situation. For example, training aids are used in tests of factual memory, for inspirational purposes, in the development of concepts, in the practice of skills, in orientation to new situations, and especially in teaching the relationships of parts of an operating machine. In fact, the range of applications seems to be limited only by the imagination, the resourcefulness, and the competence of the training personnel.

Since this amazing development of training aids is essentially the product of the thought, research, and effort of professional educators now serving as wartime officers, it is safe to predict that multi-sensory aids will be widely used in the post-war period provided that the public is willing to furnish large funds for materials and personnel. It is also obvious that multi-sensory aids are especially useful in teaching science and mathematics.

Thesis 34. *The resourceful teacher of mathematics should be given competent guidance in the production, selection, and use of slide films.*

There are good reasons for believing that a teacher of mathematics can make a significant improvement in his work by the wider use of slide films. The slide film or film strip consists of a series of "frames" that may be projected on the ordinary wall of a class room. It has the very great advantage that a single slide or frame may be held on the screen or wall for a period of time that is adequate for careful study. Fortunately film strips are not too expensive to produce nor too difficult to design by a resourceful teacher. It is conceivable that a teacher with imagination, in a favorable school situation, may be disposed to experiment with slide films. If this should happen, the total number of such teachers in the nation will be more than necessary to provide a complete set of slide films for all curriculum units that can be illuminated by this technique, if properly coordinated. The increased interest in multi-sensory aids is likely to produce a vast number of slide films—some excellent, from which the teacher must make a selection.

The Council is the appropriate organization to sponsor, guide, and coordinate the production of an excellent collection or library of slide films. The Council could promote such a program by (a) providing

[16] See *Multi-Sensory Aids in the Teaching of Mathematics*. The 18th Yearbook of The National Council of Teachers of Mathematics. The Bureau of Publications, Teachers College, 525 W. 120th St., New York 27, N. Y. Price, $2.

[17] For a more complete picture of the wide use of training aids, the reader may wish to refer to three articles by Lieutenant Commander Francis W. Noel, U.S.N.R., in the *School Executive*, February, March and April, 1944.

the necessary directions for making a slide film, (b) giving generous recognition and reasonable compensation to the person or persons creating the slide films, (c) utilizing the audience situations at regional and national meetings for the selection of the best slide films, (d) arranging for the effective distribution of its films by an organization that would operate on a low service charge and turn over a small profit on each film to the Council, and (e) giving wide publicity to the film strips sponsored by the Council.

Though the film strip seems at the moment to offer the most promising possibilities for mathematics, it is obviously not the only one of the newer aids that deserves study and experimentation. The fact that the 18th yearbook of the Council deals with the broad topic of multi-sensory aids is evidence of the importance of other aids. Questions and proposals relating to training aids deserve careful study by a special committee of the Council. The Commission, therefore, has recommended to the Board of Directors of the Council, that a standing committee on multi-sensory aids be created to study developments, to keep the council informed and up-to-date, and to make information about the best multi-sensory aids generally available to mathematics teachers.

In conclusion, let it be repeated that the foregoing Report is tentative and provisional. It does not offer final solutions, but submits a set of theses which, it is hoped, will stimulate further deliberation and discussion among the nation's educators, administrators, and teachers of mathematics. It should also be stated that the Report is the result of intensive group conferences and of much correspondence. Such discussions by local groups should resolve differences of opinion as to certain details. Absolute agreement under ordinary circumstances is not to be expected. Nevertheless most of the theses of this report were endorsed unanimously and all others were approved by a substantial majority of the members of the Commission.

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A New Approach to College Mathematics

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ARITHMETIC is the foundation of all mathematics. It took a world war to prove that fact. As a result, elementary and secondary schools are revising their mathematics programs to fit the new discovery. Meanwhile the unhappy college teacher has still to worry about the many freshmen who had their training in fundamentals in the days when computational skills were held in contempt.

Yes, arithmetic has to be taught in college. Unfortunately, besides the difficulty of giving college credit for such content there is the little matter of student attitude. College freshmen very rightly resent being subjected to drills, and to topics that are only too obviously meant for less mature minds. We refer to the so-called refresher courses and the many watered-down college algebra texts.

In fact any attempt to solve the problem by adapting elementary courses to college situations is doomed to a similar fate. The traditional approach to algebra was all right twenty years ago, but today the realistic teacher has to discard completely all that he considered important and face things as he finds them.

A New Approach

Four years ago a group of teachers at Rockhurst College began experimenting with the idea that computational skill is essential to the appreciation of algebraic symbolism. They soon realized that it was futile to try to adapt standard texts to their purposes, because all of these either envisaged students who had facility with fractions and decimals, or else completely disregarded the matter and gave exercises that avoided all but single digit whole numbers.

After stocking the library with the old texts for ready reference, new mimeographed lessons were handed out to the students daily in the classroom. This system proved very efficient because the sheets could so easily be revised and improved as new difficulties and suggestions came up in actual classroom experiences.

The result was a completely new approach to algebra through arithmetic. At first only those students were considered whose background in mathematics was deficient and who were grouped in special classes because of their showing in achievement tests. Now, all freshmen are required to follow the new system, though the classes are all very carefully grouped and many changes of individuals to the better or the poorer sections are made during the first months. Two definite discoveries are worth emphasizing:

1. Some seventy per cent of the deficient students who found mathematics almost impossible in the beginning ended the year as average college algebra students.

2. A good number of talented students who came to college with three years of high school mathematics proved quite deficient when it came to hard work on more complex problems. They tired too easily.

Before describing the content and machinery of the new course we must say a word about a fundamental assumption that colors all that is to follow. It has to do with the meaning of mathematics and the part that pure mathematics plays in the life of any educated man, including the scientist and engineer.

Mathematics for Its Own Sake

No term is more misunderstood than the word mathematics. Possibly this is because no one has ever adequately defined it, or more probably because of the inferiority complex that has suppressed any attempt on the part of the professional

mathematician to assert himself and tell the innocent teacher of the elementary phases of his subject what is expected of his pupils when they apply mathematics later. So much opprobrium has been heaped on these mathematicians during the last twenty years by so-called educators that they prefer to hide away in their secure lecture halls of graduate courses and enjoy the sincere plaudits of the few students that come to them.

Mathematics is NOT a difficult subject in the sense that it is hard to understand. What makes it seem abstruse and deep is the fact that it so often deals with complexities. But complexity is a very different thing from profundity.

This complexity is just what makes mathematics so valuable from an educational viewpoint. There is no other subject in the curriculum so well fitted for training young and old in the ability to sit down and face the details of a complex situation. This quality is quite essential to any successful human being and the very life-blood of the engineer and scientist.

In short, mathematics is educationally valuable only because it is a discipline. And it must be learned as a discipline or not learned at all

It may be objected that few can be made to like such an ascetical subject. That is not true. Many minds revel in detail work. If there are some who detest it, that is their misfortune; and it is the problem of the educator to correct this deficiency.

Understanding Mathematics

Nearly all comments and reports on the teaching of mathematics blame much of our failure on the fact that students are not made to understand what they are doing. As a matter of fact there is an over-insistence on thinking, that is, thinking without action. It is easy enough to criticize the teacher who wants his students to work by blind imitations, but a closer study of the facts shows that the products of such teachers do not so easily forget their mathematics, while the so-called "thinkers" do. Why is this?

The understanding of symbols, equations, fractions, and exponents is not brought about by long and brilliant explanations nor by discussion. In fact nearly all of this type of teaching is a waste of time, because explaining algebraic processes is much like delivering a learned lecture on cross-word puzzles. Mathematics is learned by doing, not by listening to someone explain or by watching a skilled lecturer disport himself at the blackboard.

Again, mathematics is made up of a series of simple operations. Failure to learn a mathematical idea is due to the fact that something fundamental to it was only half learned; with the result that when the student is asked to apply it to something else he becomes confused and a later teacher judges that he never understood it. For example, no amount of explaining will ever make the addition of fractions any clearer. But one who has not been thoroughly drilled in the process will seem to be the worst of dullards when suddenly asked to check even simple equations with a fractional answer.

The New Course

Hard work, then, is the motto of the new course. The first sheet is on the addition and multiplication of fractions. This is followed by twelve lessons on simple equations with fractional answers that must be checked. The lesson sheet contains little more than a form problem and series of exercises, none of them any more difficult than the sample shown. The teacher does practically no lecturing (about ten minutes a week). After quickly working the first problem on the board, he spends the remainder of the hour walking from desk to desk giving individual help and suggestions. During these "laboratory periods" the student does about one-third of the assigned work.

In the beginning there are four or even five tests a week. These tests are always

brief (two or three problems). Each problem is handed in as it is finished. Only perfect work is accepted.

This algebra is followed by a complete course in numerical trigonometry and logarithms, using first a three-place table and then a four-place one. The emphasis is on decimals and simple equations, and the meaning of symbols.

The rest of the course covers fractiona equations, quadratics, determinants, graphing, the binomial theorem, progressions and the theory of equations.

Though the matter content in the above outline seems to be much the same as that of the standard freshman course, the approach is a radical departure from tradition. To begin with, in the new approach the traditional drill in the fundamental operations is completely omitted. There is no section on the addition and multiplication of polynomials, nor any lengthy treatise on factoring. All processes are introduced through exercises on equations first in one unknown, then in two and three. Even the laws of exponents and radicals are confined to their appearances in quadratic equations, though a more thorough study of them appears before logarithms.

The second departure from "modern tradition" comes in the fractional answers, many of which are large. In this way deficiencies in arithmetic are quickly taken care of by insisting that all answers be carefully checked. An unchecked answer is looked upon as valueless simply because without checking there is no certainty whether it is right or wrong.

Conclusions

This plan may very well sound like the ebullition of a group of young teachers who think that they have made a great discovery. However that may be, the important point is that even under present ideal conditions of wartime motivations when mathematics is considered the most important subject in the curriculum, almost no improvement has been noted in either the attitude or knowledge of the student.

Too much is being said and suggested about improving teachers and providing students with better motivation. As a matter of fact this country is full of excellent teachers and no school subject has ever received such nationwide publicity. The Rockhurst experiment seems to indicate but one thing: matter-content and objectives are what need the reform.

The last hundred years have left us with a huge mass of mathematical material; still, almost none of the new matter has yet appeared in the classroom. Look for the moment at traditional courses. Nearly 60% of the content of algebra from first high to calculus is of no consequence either educationally or mathematically. Geometry is even a worse offender Recent Army and Navy courses have shown repeatedly that the essentials of Euclid's geometry can be successfully taught in two or three weeks: and at least one high school has actually substituted standard college analytic geometry in second high with a vast improvement in student interest and no loss in mathematical preparation.

Now is the time for a complete break with a doubtful tradition, and a little encouragement for those who want something slightly different from ethereal educational theories. Mathematics teaching will be improved only by pooling the knowledge of the professional mathematician and the really high class engineer with the experience and experiments of the enthusiastic teacher.

The author is in close agreement with Buell's article "Let Us Be Sensible About It." In fact much of the experiment described which appeared in The Mathematics Teacher for November 1944 had to do with the problem of the student who, due to his previous training, was continually worrying about the whys and the wherefores of things. On the other hand the ones who blindly followed instructions and worked large numbers of problems soon came to *understand* what they were doing from the sheer necessity of speeding up their work.

The Trigtractor—A Visual Aid for Teaching Trigonometry

By Edwin Eagle
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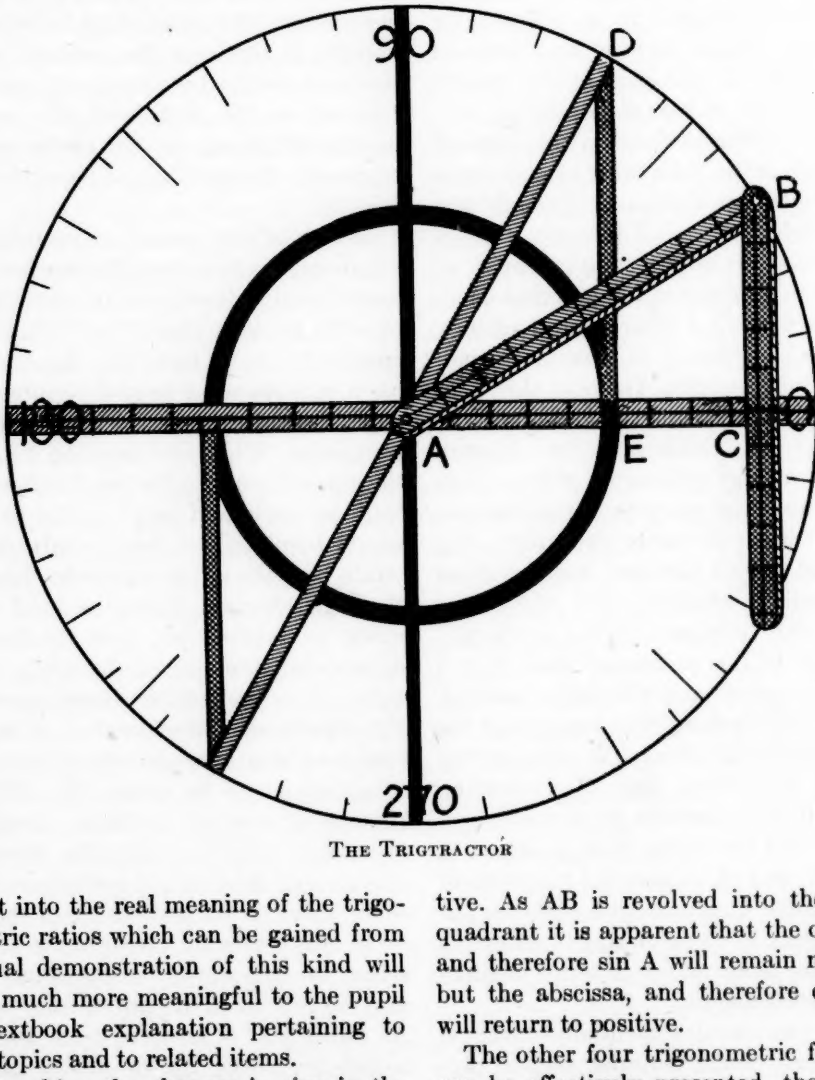
The use of the word, function, in connection with the trigonometric ratios, implies related change. It is difficult, by means of a picture, or even by a series of still pictures, to develop in the pupil's mind this idea of functional change relationship. Failure to develop this concept adequately is the basic cause of very much of the difficulties which arise throughout a trigonometry course. To many, "sin A equals opposite side over hypotenuse, or ordinate over distance," is a verbal memorization without a clear mental picture, and with the idea of functional dependence entirely lacking. To meet this difficulty in his trigonometry classes at the University of California, Dr. Merton Hill uses a large protractor with a movable radius strip carrying a movable ordinate strip at its outer extremity. The visual aid device pictured and described in this article, which for lack of a better term I call a "trigtractor," is a further development of this protractor idea. It is a real time saver and simplifies matters much in: (1) teaching clear concepts of the trigonometric functions; (2) representing functions as a single line; (3) explaining changes in the functions from positive to negative and vice-versa from quadrant to quadrant; and (4) comparing functions of angles in one quadrant with those in another, especially in expressing functions of obtuse and reflex angles in terms of functions of acute angles.

The device consists essentially of a circle of some lightweight material, such as plywood, and two narrow strips, each equal in length to the radius of the circle. As shown in the diagram, one of these is mounted at the center of the circle so that it can be revolved as a radius; the other is similarly mounted at the outer extremity of the radius strip so that it can be continuously kept vertical as the radius is revolved. This strip which is to be kept vertical to represent the ordinate is divided into tenths by prominent markings. Painted on the circle and also carrying decimal divisions are horizontal radii to represent the positive and negative abscissae.

In explaining points concerning the trigonometric functions the teacher has a continuously developing mental picture of what he is saying. The "trigtractor" enables him to be more sure that the pupil has a corresponding mental picture undergoing the same changes as the explanation progresses. When defining the functions the teacher may, with the "trigtractor," form an angle and point to the line segments involved; he may clearly demonstrate how these line segments change as the angle changes. Thus, the pupil can be made to actually *see* that the function depends on the size of the angle for its value. After the pupil has been exposed to the definitions of the functions, it may be explained that assuming the radius of the "trigtractor" to be unity, then BC, the portion of the red ordinate above the horizontal diameter, actually represents sine A, and that AC, the abscissa, represents cosine A. By making use of the divisions into tenths on the ordinate and abscissa, the sine and the cosine of any angle may be estimated to two decimal places. As radius AB is revolved from the horizontal line, AO, up to ninety degrees, with BC continuously kept vertical, the pupil can *see* cosine A decrease from one to zero and he can *see* sine A gradually increase from zero to one. By placing AB to form an angle of thirty degrees with the horizontal, and by comparing functions

of this angle with those of the painted sixty degree angle, EAD, it can be readily shown that sin 30° = cos 60°, and that cos 30° = sin 60°. This should be done when explaining that *co*sine means sine of the *com*plement. In the same way the tangent-cotangent (co-function) relationship can be demonstrated and explained. The CB will still be positive, extending upward from the green horizontal base diameter. It is similarly obvious that cosine A will become negative, for the abscissa will extend to the left. In the third quadrant the ordinate and the abscissa can be *seen* to be negative; hence both sine A and cosine A for third quadrant angles must be nega-



THE TRIGTRACTOR

tive. As AB is revolved into the fourth quadrant it is apparent that the ordinate, and therefore sin A will remain negative; but the abscissa, and therefore cosine A will return to positive.

The other four trigonometric functions can be effectively presented, though not quite so simply, on the "trigtractor." To show that the tangent can be represented as a single line, AE may be selected as the unit radius, which makes DE the tangent of the painted sixty degree angle EAD. It

insight into the real meaning of the trigonometric ratios which can be gained from a visual demonstration of this kind will make much more meaningful to the pupil the textbook explanation pertaining to these topics and to related items.

In teaching the changes in sign in the functions from quadrant to quadrant the device is equally effective. When AB is revolved into the second quadrant it is evident that sine A will remain positive when the angle becomes obtuse, for the ordinate

is convenient in this connection to point out that the tangent function is so named because DE is actually a tangent to the small circle. In this same triangle AD is the secant of A. Again, it may be pointed out that AD is a portion of a secant line; hence the name, secant, for this trigonometric ratio. By placing AB at thirty degrees and swinging CB up to a horizontal position tangent to the small circle, the cotangent and the cosecant of thirty degrees can be indicated, each represented as a single line.

For comparing angles in the second quadrant with functions of acute angles, AB may be moved to one hundred twenty degrees, or to one hundred fifty degrees as desired. Functions of these angles may then be compared with functions of the sixty degree angle EAD. Such relationships as $\sin 120° = \sin 60°$, $\cos 120° = -\cos 60°$, $\sin 150° = \cos 60°$, $\cos 150° = -\sin 60°$, etc. can be made apparent. From such demonstrations the pupil can be led to generalize concerning functions of angle A compared with those of $90° + A$ and with those of $180° - A$. The use of the "trigtractor" in extending these methods of study to angles of the other quadrants is obvious. With an approach of this kind the pupil need not memorize formulas for expressing functions of obtuse angles and reflex angles in terms of acute angles. He can see from this device what acute angle to compare the given angle with to determine the size and the sign of the required function.

The above uses of the "trigtractor" are but suggestive; throughout the course the teacher will find many times that it can be used to present vividly and at a great saving of time, explanations which can be only crudely and ineffectively presented by means of blackboard drawings. Important as this is, the chief value is probably not the fact of having immediately available a clear and striking illustration of a trigonometric principle or problem. Rather it is the possibility of actually *showing* the functions of the angle continuously changing in size, and at times in sign, as the angle changes. The clearer concepts thus developed will do much to eliminate some basic difficulties for many pupils and to make trigonometry for them more of a meaningful and logical science.

The "trigtractor" which I had made in the Taft Junior College wood shop, and which has proved very satisfactory, makes use of the following dimensions, markings, and colors. The circle is of plywood, and is ten inches in radius. For a large lecture room a larger size would be better. The vertical and horizontal diameters, and the 0, 90, 180, and 270-degree points are denoted as shown. Marked along the circle are the ten-degree divisions, and more prominently the multiples of forty-five degrees. To give prominence to abscissa readings, a green strip three fourths of an inch wide is centered on the horizontal diameter with markings at each inch, thus giving decimal divisions of the radii. The movable radius and the ordinate are wooden strips three fourths of an inch wide and also have the decimal divisions. Colored translucent plastic strips with a sharp black line down the middle would be more satisfactory than wooden strips. They are mounted on bolts with thumb screws and washers so they may be adjusted to turn freely or tightened so they will remain in any desired position. The radius strip is blue and the ordinate strip is red. The circle of five inch radius, the sixty degree angle, and the two hundred forty degree angle, together with their red ordinate lines are painted on the plywood as shown. This color arrangement makes the radius or distance blue in all cases, the ordinate lines all red, and the abscissa always green. The mentioning of the colors as explanations are given assists the pupil in following the demonstration.

The "trigtractor" is the most effective visual aid for teaching trigonometry that I have used. If effectively designed and well constructed, with sharply contrasting colors, it has all the advantages of lantern slides, wall charts, or carefully and color-

fully drawn blackboard sketches. In addition to this it has all the advantages of motion picture demonstrations with slow motion, stop, and reverse control. Most important of all, it is more readily available than these other visual aids, and it is more adaptable to a variety of principles and problems that repeatedly occur in a trigonometry class.

Since writing this article the author has found, from suggestions of other teachers and by using the trigtractor that its effectiveness is definitely increased by including a square circumscribed about the circle and by lengthening the radius strip to extend beyond the circle. This makes it possible to directly portray the tangent, cotangent, secant, and cosecant on the unit circle, and to observe them change as the angle changes.

Some More Thoughts on Placing the Decimal Point in Quotients

By J. T. Johnson
Chicago Teachers College, Chicago, Ill.

The article on *Some Thoughts on Placing the Decimal Point in Quotients* which appeared in the February issue of the *Teacher* calls for a continuation of some more thoughts as it leaves the reader somewhat in doubt and it does not give all of the weaknesses of the so-called logical method.

For those who did not read the article by Mr. Brown, he discusses the pros and cons of the two methods of pointing off the quotient in division of decimals—whether to put the decimal point in first or last.

The place where the subtraction method (which Mr. Brown calls the logical method) breaks down is in the case where there are more decimal places in the divisor than in the dividend and also where there are no decimal points in either divisor or dividend but the divisor a whole number greater than the whole number dividend as in this example:

$$7\overline{)3.}$$

This is very common and occurs whenever a common fraction is changed to a decimal. In this case what generally happens with a pupil using the subtraction method is that he adds a couple of zeros to the dividend at first as follows:

$$\begin{array}{r} 428 \\ 7\overline{)3.00} \\ \underline{28} \\ 20 \\ \underline{14} \\ 60 \\ \underline{56} \end{array}$$

then he may continue or stop after three divisions are made and apply his rule which says, subtract the number of decimal places in the divisor from those in the dividend, in this case $2-0$ and he points off 2 places in the quotient, 4.28 and gets his answer wrong. Why? because he forgot to place a zero in the dividend every time he divided.

That error was so common among students (the writer has used this method for a long time) that the new method, called by Brown the caret method, was inaugurated to avoid that error. Witness the advantages when the pupil puts his decimal point in the quotient first, as

$$7\overline{)3.0.}$$

Then he divides and carries his division to as many places as is called for. If two decimal places are called for, he continues until his quotient shows two decimal places regardless of how many he has in the dividend as follows:

$$\begin{array}{r} .42 \\ 7\overline{)3.0} \\ \underline{28} \\ 20 \\ \underline{14} \end{array}$$

Whenever he stops dividing his decimal point is in the right place. He does not need to put a zero in the dividend every time he divides as the pupil of the subtraction method should but forgot to do. This method has been used now so long that it came as a surprise to read that there was any question about it.

In teaching elementary pupils first division of decimals we begin with the easiest phase, that of dividing a decimal by a whole number, as in this example,

$$4\overline{).8}^{\;.2}$$

and the decimal point is placed directly over the decimal point in the dividend. This is easily explained because if you are dividing tenths by 4 you will get tenths just as dividing apples by 4 you will get apples. There is no violation of logic when later the pupil proceeds with more difficult examples and makes the divisor a whole number first and then proceeds as he was taught from the beginning. It would seem then that the logic is as good in the caret method as it is in the subtraction method.

One wonders at statements such as the following;

"The caret method, however, may be taught as a trick, a short cut which the child may use without understanding why it works."

It does not have to be taught as a trick. On the other hand, cannot the subtraction method likewise be taught as a trick?

"In other words, even if the caret method is more accurate in school-room situations, it does not follow that it will be equally accurate in out-of-school situations."

This statement surprises us in three ways. First, in that it is admitted that the caret method might be more accurate. Second, in that a method which is more accurate in school should not be more accurate outside. Third, that business men use the caret method with the caret left out.

"Furthermore, the subtraction method, if mastered by the student, will serve to locate the decimal point properly in any situation involving division by decimals. This is not true of the caret method."

The first statement above may be true but the second one certainly is not. If either method is *mastered* it will serve the purpose equally well, the only difference being that it takes longer to master the subtraction method and errors persist in it long after the teacher thinks it has been mastered.

Readers of these articles may say we have proved nothing. True, we have not. If the question is serious enough it should be submitted to a scientific test. We now have the necessary statistical technique wherewith to do this, and this question lends itself admirably to the differentiated method of statistical treatment. Some one searching for a master's thesis topic should carry this through.

Several hundred pupils of the same grade and mental age should be selected from each method. Each group should be given a test in ordinary division with time and errors noted for each group. Then these same groups should be given the same division examples with decimal points involved and time and errors again noted.

As division of decimals is always tied up with ordinary division, the above method will isolate the ability, "pointing off the quotient" for measurement, and the differences in the excesses of time and errors in the two tests will indicate the time and errors due to the pointing off alone. These can then be compared.

This does not take into account the initial time used in learning each method. That would constitute another experiment.

Let us use the scientific method in settling our questions.

EDITORIAL

What Next?

THIS ISSUE of THE MATHEMATICS TEACHER contains *The Second Report of the Commission on Post-War Plans* of The National Council of Teachers of Mathematics. This Commission has been working now for more than a year. Whether the National Council will continue the work has not been determined. Considering the time that the individual members have been able to give to this work and the limited amount of money at its disposal, the Commission has rendered a great service not only to the teachers of mathematics, but also to general education. To be sure this Report, like all the others that have preceded it, does not give definite help on every possible issue. In fact that is not desirable or necessary at this time. For example, this new Commission did not give as much help to teachers of geometry as it did to teachers of algebra and general mathematics. It does not say whether the number of text-book theorems should be reduced in line with Pickett's research,[1] whether the present emphasis on practical applications, particularly those dealing with navigation, is being overdone, whether informal solid geometry should be made an integral part of the course with or without formal proof, whether the number of original exercises had been unduly increased in view of the objective to be obtained, or whether it was advisable to teach the pupil how the study of geometry might help him better to think about many matters in life situations, i.e. the importance of teaching the pupil something about the different types of thinking. In other words, they had to leave some things unsaid even though the Commission itself may have considered them. Moreover, nothing was said in the report about counselling or guidance of pupils, although the Commission gave a great deal of time and thought to this matter.

If and when this Commission continues to function further, attention will undoubtedly be given to important matters which the second report could not include.

When one considers the notable reports of such Commissions in the last twenty-five years[2] one can see what a wealth of

[1] Pickett, Hale, ***An Analysis of Proofs and Solutions of Exercises Used in Plane Geometry Tests***, Teachers College, Columbia University Contributions to Education, No. 747.

[2] The significant reports in the teaching of mathematics in the last twenty-five years are as follows:

1. *The Reorganization of Mathematics in Secondary Education*—A Report by The National Committee on Mathematical Requirements under the auspices of The Mathematical Association of America. Now out of print, but still a most helpful reference.
2. *The Place of Mathematics in Secondary Education*—A Final Report of the Joint Commission of The Mathematical Association of America and The National Council of Teachers of Mathematics. This report was published as The Fifteenth Yearbook of The National Council of Teachers of Mathematics. The Bureau of Publications, Teachers College, 525 W. 120 St., New York 27, N. Y. Price $1.75 postpaid.
3. *Mathematics in General Education*. The final report of the Committee on "The Function of Mathematics in General Education" for the Commission on The Secondary School Curriculum of The Progressive Education Association. D. Appleton-Century Co. New York City.
4. "Pre-Induction Courses in Mathematics." THE MATHEMATICS TEACHER, March 1943. This report has been submitted to and approved by: The National Policy Committee for High-School Victory Corps; The Civilian Pre-Induction Training Branch, Industrial Personnel Division, Services of Supply, War Department; The Training Division Bureau of Naval Personnel, Navy Department, The Civil Aeronautics Administration, Department of Commerce. Reprints of this report may be had at 10¢ each postpaid from THE MATHEMATICS TEACHER, 525 W. 120 St., New York 27, N. Y. Larger quantities may be had at a reduction.

suggestions and advice have been given by these various groups. They should be made the basis for immediate action by The National Council. Action is what is needed now. We can no longer attempt to solve all of the problems of the schools and particularly those in regard to the reorganization of mathematics by traditional methods. Whatever can be learned from the recent war experience is all to the good, and let us hope there may be a great deal. Let us also hope that more and better prepared teachers will be available and at salaries that will enable them to have a decent living. However, no matter what happens, we should not expect miracles and it is clear to those who understand the present situation that in many places better teachers, higher salaries, and more money for better books, equipment and the like may not be forthcoming. This means only one thing, namely, that teachers who remain in the classroom must somehow learn to teach in better and more interesting ways many of the things that are now in the course of study. This need not be discouraging, for this can be done, but if, and only if, teachers are inspired to carry on the work as outlined by all the Commissions which have tried so hard to be helpful.

We still have too many teachers of mathematics who do not subscribe to THE MATHEMATICS TEACHER, who do not read the Yearbooks of The National Council of Teachers of Mathematics, both of which are great sources of help and inspiration. Many important devices, models, and some of the more important newer items of class room equipment are inexpensive if a teacher is ingenious, interested and resourceful. The new (Eighteenth) Yearbook is an excellent sourcebook of helpful multisensory aids.

We believe that now is the time for The National Council of Teachers of Mathematics to continue to pioneer. If more commissions are needed let's have them, but let us also show that we understand and appreciate what such groups have done for mathematics, by united group action all over the country.—W.D.R.

5. "Essential Mathematics for Minimum Army Needs."—A final report by a committee of the U. S. Office of Education working in conjunction with the Civilian Pre-Induction Training Branch and The National Council of Teachers of Mathematics. This report was published in THE MATHEMATICS TEACHER for October 1943, pp. 243–282. Reprints of this report may be had from THE MATHEMATICS TEACHER, 525 W. 120 St., New York 27, N. Y. for 15¢ each postpaid.
6. "The First Report of the Commission on Post-War Plans."—THE MATHEMATICS Teacher, May 1944, pp. 226–232. At a meeting in New York City, February 25, 1944, the Board of Directors of the National Council of Teachers of Mathematics created a commission to plan mathematics programs for secondary schools in the post-war period. This brief preliminary report represents a first step in the work of the Council's new Commission. Reprints of this report may be had from THE MATHEMATICS TEACHER, 525 W. 120 St., New York 27, N. Y. for 10¢ each postpaid.

Silver Burdett Has a Birthday

Silver Burdett observes a gala occasion this year as the company reaches its sixtieth birthday. Edgar O. Silver, the founder, would scarcely recognize the offspring of the humble publishing business he set up in Boston in 1885. His list of one item, THE NORMAL MUSIC COURSE, has grown and expanded to include basic texts in the major subjects on both primary and secondary school levels, and Silver Burdett Company has emerged as one of the country's leading textbook publishing houses with offices in New York, Chicago, and San Francisco.—W.D.R.

IN OTHER PERIODICALS

By NATHAN LAZAR
Midwood High School, Brooklyn 10, New York

American Mathematical Monthly
January 1945, Vol. 52, No. 1

1. Jonah, H. F. S. and Keller M. W., "Some Variations of the Multiple-Choice Questions," pp. 1–6.
2. Hempel, C. G., "Geometry and Empirical Science," pp. 7–17.
3. Martin, W. T., "Functions of Several Complex Variables," pp. 17–27.
4. Menger, Karl, "Methods of Presenting e and π," pp. 28–33.
5. Robinson, H. A., "A Problem of Regions," pp. 33–34.
6. Eves, Howard, "Feuerbach's Theorem by 'Mean Position'," pp. 35–36.
7. War Information (Edited by C. V. Newsom); Aid to Libraries in War Areas; Education of Students from War-torn Countries; Academic Standards for V-12 Officer Candidates; Civil Service Appointments for Mathematicians.

Bulletin of the Kansas Association of Teachers of Mathematics
February 1945, Vol. 19, No. 3.

1. "Post-War Planning for Mathematics Education," pp. 35–36.
2. Janes, W. C., "Mixtures" (a radio talk given over KSAC), pp. 36–38.
3. Zant, James H., "Jobs Available for Mathematics Students," pp. 38–40.
4. Ulmer, Gilbert, "Tests for Critical Thinking" p. 40.
5. Betz, William, "Next Steps in Education in the Teaching of Mathematics," pp. 41–48.
6. Neuschwanger, A., "Salina Course of Study for First Year Algebra," p. 48.

The Mathematical Gazette
December 1944, Vol. 28, no. 282.

1. Hansel, C. W., "The Use of a Mirror in the Teaching of Mathematics," pp. 165–169.
2. Gibbins, N. M., "Infinite Series for Fifth-Formers," pp. 171–172.
3. Atkinson, E. J., "Simple Harmonic Motion Examined," pp. 173–175.
4. Sandon, F., "Some Principles and Practical Points in Preparing Teaching Time-Tables," pp. 176–180.
5. Mathematical Notes:
1746. Kailasamaiyer, N., "A Proof of Pythagoras' Theorem";
1747. "On Note 1680";
1748. G. A. C. "Pythagoras' Theorem";
1749. Weikersheimer, S., "The Fourth Case of Congruence";
1750. S. L. G., "The Angular Momentum of a Lamina";
1751. Lawrence, B. E., "Pythagoras and an Extension";
1752. Roth, L., "The Solution of Linear Differential Equations of the Second Order";
1753. Deans, J., "The Asymptotes of the Hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$;
1754. Wiseman, C. L., "The Sign of ρ and Related Topics";
1755. Burnham, C. E. A., "On Phase Angle";
1756. "Howlers";
1757. Todd, H., "Reduction of the General Conic when $h^2 \pm ab$;
1758. Fisher, C. H., "The Director Circle of the General Conic";
1759. Green, H. Gwynedd, "The Circle of Curvature in Cartesian Coordinates";
1760. F. W. K., on note 1669;
1761. F. W. K., "The Tangent to a Conic";
1762. F. W. K., "On Pascal's Triangle";
1763. Lockwood, E. H., "Lessons on Two-Dimensional Vectors";
1764. E. H. N., "Inversion of Order in the Second Derivative";
1765. E. H. N., "Rough Pegs, and a Problem in Algebra Hanging on Them";
1766. Joint, W. Howard, "The Trigonometric Ratios of Angles Greater than 90";
1767. Joint, W. Howard, "Cycles";
1768. Chari, V. T., "Irrational Numbers";
1769. Hodgetts, W. J., "Feuerbach's Theorem";
1770. Hodgetts, W. J., "A Useful Lemma and Pascal's Theorem";
1771. Tuckey, C. O., "The Iso Triangle";
1772. Tuckey, C. O., "$2\pi r$ or $2\pi r^2$";
1773. Clark, Edward V., "The Construction of Logarithm Tables";
1774. Carmody, Ernest P., "A Vulgar Story";
1775. Parks, W. A., "On the Teaching of Logarithms";
1776. Birch, R. H., "A Note on Logarithms";
1777. Birch, R. H., "On Note 1741";
1778. R. L. G., "Ferry Puzzles";
1779. N. M. G., "A Converse";
1780. Gibbins, N. M., "A Close Approximation to π";
1781. Gibbins, N. M., "Application of the Newton Approximation to the mth root of N";
1782. Egan, M. F., "Notes on Matrices";
1783. Nutbourne, A. W., "Two Original Proofs";
1784. Spooner, C. R., "The Logarithmic and Exponential Properties as Particular Cases of General Relations";
1785. Simpson, Harold, "The Invariant of Three Conics";
1786. Williamson, R. S., "Some Suggestions for School Work, Derived from Ancient Egyptian Mathematics";
1787. Robertson, C., "Extension of Tables of Squares";

1788. Peacock, R. H., "Concerning Note 1717";
1789. C. W. H., "A Higher Certificate Syllabus of Mathematics for Students of Physics, Chemistry, Mathematics";
1790. Sandom, Frank, "Random Sampling Numbers and the Schoolmaster";
1791. Watson, G. H., "On a Functional Equation";
1792. G. N. W., "On Pascal's Theorem Again."

School Science and Mathematics
March 1945, Vol. 45, No. 3.

1. Struk, Adrian, "Drawing with Ruler and Paper," pp. 211–214.
2. Jamison, G. H., "The Preparation of Teachers of General Mathematics," pp. 249–256.
3. Jerbert, A. J., "Square Root," pp. 265–272.
4. Hartung, Maurice L., "Attitudes Toward the Mathematics Curriculum and Post-War Planning," pp. 273–278.
5. Breiland, John G., "Mathematics in Weather Forecasting," pp. 279–282.

Miscellaneous

1. Armistead, N. G., "From 1 to 10," *Grade Teacher*, 62: 28+, March 1945.
2. Bacon, E., "Modern Objective Tests: Arithmetic" *Grade Teacher*, 62: 8, March 1945.
3. Blumberg, P. S., "Mathematics Versus the 6,000,000: a Reply to F. L. Wren," *Clearing House*, 19: 315–316, January 1945.
4. Hawkins, G. E., "Selected References on Secondary-School Instruction: Mathematics (continued)" *School Review*, 53: 114–115, February 1945.
5. Russel, M. B., "Modern Objective Tests: a Number Test," *Grade Teacher*, 62: 6, February 1945.
6. Orleans, Joseph B., "High School Mathematics Today," *Secondary Education*, 12: 1–3, November 1944.
7. Rosenberg, R. R., "Diagnostic Tests in Business Mathematics," *Business Education World*, 25: 277–278, January 1945.
8. Smith, H. L., and others, "One Hundred Fifty Years of Arithmetic Test Books," *Bulletin of the School of Education, Indiana University*, 21: 1–149, January 1945.
9. Stoner, P. W., "Trends in the Organization of Mathematics," *California Journal of Secondary Education*, 20: 57–60, January 1945.

NEWS NOTES

MATHEMATICS INSTITUTE
Duke University

The Mathematics Institute will hold its fifth annual session at Duke University from July 3 to July 12. The General theme of the Institute this year will be *Applications of Mathematics to Science, Engineering, Industry, and Business.*

In the past four summers more than 300 teachers from 18 states have attended the Institute. The purpose of the Institute is to bring together alert teachers of mathematics in grades 7–12 to study intensively the problems of common interest, and to learn new uses of mathematics. Open discussion is a feature of the Institute.

The work of the Institute will center around the Mathematics Laboratory. The Laboratory contains: a large collection of recent text books, a "Model Library" for junior and senior high schools, instruments, models, drawings, charts, maps, curriculum studies, reports, portraits of mathematicians.

Teachers are urged to bring to the Institute student-made instruments, models, drawings, etc., to place on exhibit in the Mathematics Laboratory during the Institute.

The registration fee is $2.00, board and room (double) is $2.50 per day, and $3.00 for single room.

W. W. Rankin
Director of Mathematics Institute
Director of Mathematics Laboratory

Program

Tuesday, July 3

10:00–4:00 P.M. Registration, West Duke Building

DAILY SCHEDULE

(All meetings will be held in West Duke Building, East Campus)

Tuesday, July 3

4:00–5:30 P.M.
Presiding: Professor W. W. Rankin, Duke University
"Mathematics in Action"—Professor W. J. Seeley, Duke University, Electrical Engineering
"Multisensory Aids in the Study of Mathematics"—Professor Ruth Stokes, Winthrop College, Mathematics Department
Discussion:

8:30–10:00 P.M.
Presiding: Miss Veryl Schult, Supervisor Secondary Mathematics, Washington, D. C.
"Mathematics in Economics"—Professor J. J. Gergen, Duke University, Chairman Mathematics Department
"A Social Scientist's Need for Mathematics"—Professor Hornell Hart, Duke University, Sociology Department
Discussion:

Wednesday, July 4

9:30–11:00 A.M.
Presiding: Professor H. F. Munch, University of North Carolina
"Construction and Use of Mathematical Models"—Professor Ruth Stokes, Winthrop College
"The Laboratory Method of Studying Mathematics"—Professor W. W. Rankin, Duke University, Mathematics Department
Discussion:

11:15–12:00 Noon
Committee Appointments, Study Groups Organized
Study Groups will be organized in:

1. Visual Aids in the Study of Mathematics, Advisor—Professor Ruth Stokes
2. Collecting and Analyzing Statistical Data, Advisor—Professor Douglas E. Scates, Duke University
3. Collecting and Using Library and Laboratory Materials, Advisor—Professor W. W. Rankin
4. Post-War Courses in Mathematics, Advisor—Miss Veryl Schult

Other Study Groups will be organized if needed.

Wednesday, July 4

8:30–10:00 P.M.
Presiding: Professor W. G. McGavock, Davidson College
"Some Applications of Mathematics to Designing Airplanes"—Mr. C. Bowie, Mathematician, Glen L. Martin Co.
"An Economist Looks at Mathematics"—Professor J. J. Spengler, Duke University, Economics Department
Discussion:

Thursday, July 5

9:30–11:00 A.M.
Presiding: Miss Anne Corry, Central High School, Charlotte
"Graphical Analysis of Community, State and National Data"—Professor Gertrude Cox, N. C. State College, Statistics Dept.
"Consumer Mathematics in Junior and Senior High Schools"—Miss Veryl Schult, Supervisor Secondary Mathematics, Washington D. C.
Discussion:

11:15–12:00 Noon
"Demonstration of Computing Machines"—Mr. E. S. Cabiniss, Monroe Calculating Machine Company

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8:30–11:00 P.M., Party, University House, Chapel Hill St.
Presiding: Dean Alice Baldwin, Duke University
Address: Dr. Clyde A. Erwin, Supt. Public Instruction, North Carolina
"Work of the Policy Commission"—Professor F. L. Wren, Peabody College, President of The National Council of Teachers of Mathematics
Reception: Honoring, Dr. Clyde A. Erwin, Dr. R. L. Flowers, Dr. W. H. Wannamaker, Dean Alice Baldwin, Mr. C. Bowie, Dean A. W. Hobbs, Dr. Helen Barton, Miss Veryl Schult, Miss Olive Smith, Dr. J. J. Gergen, Miss Laura Efird, Dr. H. A. Fischer, Dr. Holland Holton, Professor E. L. Mackie

Friday, July 6

9:30–11:00 A.M.
Presiding: Miss Margaret Ricks, Rocky Mount High School
"Applications of Geometry to Art and Industry"—Mrs. Ruth Lane, Woodrow Wilson High School, Washington, D. C.
"Modern Geometry as Related to Plane Geometry in High School"—Professor E. F. Canaday, Meredith College, Mathematics Department
Discussion:

11:15–12:00 Noon
Study Groups

8:30–10:00 P.M.
Presiding: Professor Helen Barton, N. C. Women's College
"Geometry in Airplane Designing"—Mr. C. Bowie, Glenn L. Martin Co.
"The Complex Number Actually at Work"—Professor Otto Meier, Duke University, Electrical Engineering
Discussion:

Saturday, July 7

9:30–11:00 A.M.
Presiding: Mrs. Rosalie Elliott, Durham High School
"Applications of Geometry to Art and Industry with Color Combinations"—Mrs. Ruth Lane, Woodrow Wilson High School, Washington, D. C.
"The Use of the Calculating Machine in Teaching Mathematics"—Mr. E. S. Cabiniss, Monroe Calculating Machine Company
Discussion:

11:15–12:00 Noon
Study Groups

Sunday, July 8

5:00 P.M., Tea, Professor Rankin's Home, 1011 Gloria Ave.
Honoring Miss Veryl Schult, Mr. F. E. Goddard, Mrs. Ruth Lane, Professor W. J. Seeley, Miss Margaret Ricks, Professor R. S. Wilbur, Professor Holland Holton, Professor J. W. Lasley, Professor Gertrude Cox, Dean Herbert Hering, Dean W. H. Hall

Monday, July 9

9:30–11:00 A.M.
Presiding: Mr. F. L. DeBruyne, Durham High School
"Survey of Educational Qualifications Required for Industry"—Mr. F. E. Goddard, Aerodynamic Research Engineer, Glenn L. Martin Co.
"The Text Book—Plus"—Miss Veryl Schult, Supervisor of Secondary Mathematics, Washington, D. C.
Discussion:

11:15–12:00 Noon
Study Groups

8:30–10:00 P.M.
Presiding: Professor R. S. Wilbur, Duke University

"The Mathematical Problems in Aerodynamics, Air Flutter and Dynamic Stability"—Mr. F. E. Goddard, Aerodynamic Research Engineer, Glenn L. Martin Co.
"Principles Involved in Diagnostic and Remedial Procedures"—Professor W. A. Brownell, Duke University, Educational Psychology
Discussion:

Tuesday, July 10

9:30–11:30 A.M.
Field Work—Demonstrations of Transit, Plane Table, Level, Sextant
Professors H. C. Bird, A. E. Palmer, J. W. Williams

11:45–12:30 P.M.
Study Groups

8:30–10:00 P.M.
Presiding: Miss Laura Efird, Raleigh High School
"Statistics on the Junior High School Level"—Miss Veryl Schult, Washington, D. C.
"Statistics on the Senior High School Level"—Professor Gertrude Cox, N. C. State College, Statistics Department
Discussion:

Wednesday, July 11

9:30–11:00 A.M.
Presiding: Miss Olive Smith, Winston-Salem High School
"In-Service Training of Teachers"—Miss Veryl Schult, Washington, D. C.
Professor H. F. Munch, University of North Carolina
Professor W. W. Rankin, Duke University
"Scale Drawing in the Study of Mathematics"—Professor K. B. Patterson, Duke University, Mathematics Department
Discussion:

11:15–12:00 Noon
Study Groups

8:30–10:00 P.M.
Presiding: Professor F. G. Dressel, Duke University
"Some Applications of Mathematics to Instruments, and Mechanisms"—Professors W. A Hinton and E. S. Theiss, Duke University, Mechanigal Eingineering
Discussion:

Thursday, July 12

9:30–11:00 A.M.
Presiding: Professor C. G. Mumbord, N. C. State College
"The Laws of Nature"—Professor Willard Berry, Duke University, Geology Department
Professor C. R. Vail, Duke University, Electrical Engineering
Discussion:

11:15–12:30 P.M.
Final Reports of Study Groups and Committees

8:30–9:30 P.M.
Presiding: Miss Veryl Schult, Washington, D. C.
"The Mathematics of Communication"—Professor W. J. Seeley, Duke University, Electrical Engineering

9:45—Curfew
Water Melon Party, Professor Rankin's Home, 1011 Gloria Ave.

The Sixth Meeting of the Men's Mathematics Club of Chicago and Metropolitan Area was held on Friday, April 20 at the Central Y.M.C.A. Dr. E. P. Northrop of the University of Chicago spoke on "Mathematics in Liberal Education" and Mr. J. A. Nyberg of Hyde Park High School spoke on "A Pre-induction Course in Mathematics for Seniors."

Miss Emma McGaughey of Mentor, Ohio, writes "I know of no professional organization which brings as much instruction and inspiration as your publication. For years I have been trying to get more mathematics teachers in this country to subscribe, but with little success."

We have some difficulty in understanding why some teachers think that THE MATHEMATICS TEACHER is so helpful and others are so uninterested considering what the needs of many such teachers really are—Editor.

D. C. HEATH AND COMPANY this year will celebrate sixty years of publishing. Late in 1885 the publishing firm of Ginn and Heath was dissolved and the new firm, established by Daniel Collamore Heath, started on its way with thirteen books and eleven pamphlets. These were chiefly in science and modern languages, two subjects that Mr. Heath had the vision to anticipate would play an important part in future school curriculums. In a recent interview, Mr. Dudley R. Cowles, president of D. C. Heath and Company, said, "We were fortunate that in our early years our steps were guided by men who as experienced educators believed strongly that an important part of their business was to advance the cause of American education by making as good books as it was possible for us to make. That purpose has remained a guide to the Company ever since. During the last sixty years Heath has pioneered in new fields, and recently has developed a large and strong list in the elementary field. In the high school and college fields Heath is going vigorously ahead to keep abreast of the many educational changes that a changing world demands. We have produced an increasing number of texts meeting educational requirements so closely that several of our series have been and are being used by the millions. It has always been our purpose to watch and understand the trends in education and to anticipate if possible or at least to meet promptly the educators' demands for textbooks as tools to carry forward their programs.

"Sixty," Mr. Cowles said, "is a fine age. We are old enough to profit by our experience; young enough to look ahead with enthusiasm, to redouble our efforts, and to do our share in meeting the new and difficult demands the postwar world will make on American schools and American publishers."